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Abstract

Statistics for testing different hypotheses in the Growth and Extended Growth Curve models are proposed. The tests are constructed using the restricted maximum likelihood approach followed by an estimated likelihood ratio. They are all functions of the residuals in the respective models. This property makes them more reasonable and natural and gives the advantage of being easy to apply. Moreover, the tests have resemblance with the Lawley-Hotelling's trace test for the classical MANOVA model. We interpret the tests in accordance with the interpretation of the residuals of the models. We also show that the distribution of the tests under the corresponding null hypotheses does not depend on the unknown covariance matrix. Conditional and unconditional expectations for the tests are given.

Keywords: Estimated Likelihood, Extended Growth Curve Model, Growth Curve Model, Residuals, Restricted Likelihood.

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1. Introduction

Fitting a model to data is an important part of understanding different phenomena around us. Models are used in predicting and making important decisions. One can follow two approaches in constructing a statistical model. The first approach is to assume a predefined model and check if the constructed model describes the situation well. The second one is to fit a model based on the observed data. In either case we need to assess the constructed model and see how well it fits to the data. The most common and natural approach for doing that is by looking at differences between observed and estimated values, i.e., by looking at the residuals which represent what is left unexplained after fitting a model.

In univariate model fitting problems, the resulting residuals are also univariate and it is relatively easy to examine them. In fact, there has been many discussions regarding the residuals and many different types of residuals have been defined and studied, see for example Sen & Srivstava (1990). However, in the classical multivariate case, there has been few studies regarding the residuals although most tests which have been proposed for this model are in some way functions of the residuals. See Srivastava & Khatri (1979) for some proposed tests.

When it comes to the Growth Curve Model (GMANOVA model), the model has more structure on the mean than the ordinary MANOVA model which makes things even more complicated. The Growth Curve Model was introduced by Potthoff & Roy (1964) and studied by many authors including Rao (1965), Khatri (1966) and von Rosen (1989). There is a book by Kshirsagar & Smith (1995), and for a review of the model we refer to von Rosen (1991). The model is also treated in Kollo & von Rosen (2005).

Definition 1.1. Let $X : p \times n$ and $B : q \times k$ be the observation and parameter matrices, respectively, and let $A : p \times q$ and $C : k \times n$ be the within and between individual design matrices, respectively. Suppose that $q \leq p$ and $\rho(C) + p \leq n$, where $\rho(.)$ denotes the rank of a matrix. The Growth Curve model is given by

$$X = ABC + \epsilon, \tag{1.1}$$

where the columns of ϵ are assumed to be independently p-variate normally distributed with mean zero and an unknown positive definite covariance matrix Σ.

Inspired from univariate models diagnostics for the multivariate models have been studied, and ordinary residuals have been used as diagnostic tools for validating the models. However, it is only recently that diagnostics for the Growth Curve Model has been considered, see for example Liski (1991), Pan

& Fang (1995, 1996) and von Rosen (1995). There is a book by Pan & Fang (2002) about statistical diagnostics for the Growth Curve Model which also gives a very good background about the model.

Although, diagnostics for the Growth Curve Model has been considered, there has been no studies connecting those studies with the residuals in the model. Moreover, due to the bilinear structure in the model the ordinary residuals in the model have different components. Therefore, one needs to investigate those components if one wants to develop diagnostic tools based on the residuals in the model. Residuals for model, taking the bilinear structure into consideration, were defined by von Rosen (1995):

$$
\begin{aligned}
R_{g1} &= A(A'S^{-1}A)^{-}A'S^{-1}X(I - C'(CC')^{-}C); && (1.2)\\
R_{g2} &= (I - A(A'S^{-1}A)^{-}A'S^{-1})X(I - C'(CC')^{-}C); && (1.3)\\
R_{g3} &= (I - A(A'S^{-1}A)^{-}A'S^{-1})XC'(CC')^{-}C, && (1.4)
\end{aligned}
$$

where $S = X(I - C'(CC')^{-}C)X'$. Observe that there are three different residuals.

The result was then extended to a special case of the Extended Growth Curve Model by Seid Hamid & von Rosen (2005a). Their approach is somewhat different and seems more natural than the one in von Rosen (1995).

The Extended Growth Curve Model was introduced by von Rosen (1989) although a canonical form was considered by Banken (1984). A special case of the model which was considered by Seid Hamid & von Rosen (2005a) together with the assumptions made in their paper is given below. We have used the same notations as in that paper except that we have added "e" in the subscripts for the residuals to identify them from those in the Growth Curve Model where we have used "g" instead.

Definition 1.2. Let $X : p \times n$, $A_1 : p \times q_1$, $A_2 : p \times q_2$, $B_1 : q_1 \times k_1$, $B_2 : q_2 \times k_2$, $C_1 : k_1 \times n$, $C_2 : k_2 \times n$. Suppose that $q_1, q_2 \leq p$, $\rho(C_1) + p \leq n$ and $\mathcal{C}(C_2') \subseteq \mathcal{C}(C_1')$, where $\rho(.)$ and $\mathcal{C}(.)$ represent the rank and the column space of a matrix, respectively. Then the Extended Growth Curve Model is defined by

$$X = A_1B_1C_1 + A_2B_2C_2 + \epsilon, \qquad (1.5)$$

where ϵ is as in Definition 1.1.

In addition to the nested subspace condition included in the above definition, it was assumed that there were two groups, Group I and Group II. It was also supposed that the individuals in Groups I and II follow linear and quadratic mean structures, respectively. Moreover, it was assumed that the growth curve of the individuals in the second group consists of a linear term in addition to the intercept and quadratic terms. Note that it is possible to have subgroups under both groups.

In Seid Hamid & von Rosen (2005a), four residuals were defined for this model by decomposing the space orthogonal to the space generated by the design matrices. Moreover, some properties of the residuals and a natural interpretation together with an explanation as to how one can use them for validating the fitted model were also given there. The residuals are presented below:

$$
\begin{aligned}
R_{e1} &= (I - T_1)X(I - C_1'(C_1C_1')^- C_1); && (1.6)\\
R_{e2} &= T_1X(I - C_1'(C_1C_1')^- C_1); && (1.7)\\
R_{e3} &= T_1X(C_1'(C_1C_1')^- C_1 - C_2'(C_2C_2')^- C_2); && (1.8)\\
R_{e4} &= (T_1 + T_2 - I)XC_2'(C_2C_2')^- C_2, && (1.9)
\end{aligned}
$$

where

$$
\begin{aligned}
T_1 &= I - A_1(A_1'S_1^{-1}A_1)^- A_1'S_1^{-1}; && (1.10)\\
T_2 &= I - T_1A_2(A_2'T_1'S_2^{-1}T_1A_2)^- A_2'T_1'S_2^{-1}; && (1.11)\\
S_1 &= X(I - C_1'(C_1C_1')^- C_1)X'; && (1.12)\\
S_2 &= S_1 + T_1XC_1'(C_1C_1')^- C_1(I - C_2'(C_2C_2')^- C_2) && (1.13)\\
&\quad \times C_1'(C_1C_1')^- C_1X'T_1'.
\end{aligned}
$$

There has been some studies regarding hypotheses tests in the Growth Curve Model given in Definition 1.1. However, there has not been any studies what so ever connecting the tests in this model with the residuals.

This paper is a continuation of the papers by von Rosen (1995) and Seid Hamid & von Rosen (2005a) where it was suggested that one can use the residuals for validating the fitted model. As to our knowledge, no one else has tried to use the residuals in this model to check the adequacy of the model although it is the most convenient and natural way of doing it.

Here we take the next step and use the residuals for validating the model via hypothesis testing. We consider the two models given in Definitions 1.1 and 1.2. The aim is to formulate reasonable and practical hypotheses, say for example, checking the adequacy of the model. Then, we construct test statistics using the restricted followed by the estimated likelihood approaches. That is by maximizing a part of the likelihood function which is independent of the parameters of interest in order to estimate the covariance matrix, Σ. The estimated likelihood which is obtained by replacing Σ by its estimator is then maximized under H_o and $H_o \cup H_1$, where H_o and H_1 are the null and alternative hypotheses, respectively. More details about the restricted maximum likelihood approach can be found in Patterson & Thompson (1971) and Searle et al. (1992). For the Growth Curve Model, the method has been discussed in Pan & Fang (2002).

As expected, the tests turn out to be a function of the residuals defined in von Rosen (1995) and Seid Hamid & von Rosen (2005a) for the models given in Definitions 1.1 and 1.2, respectively. This desirable characteristic of the tests makes them reasonable and natural and gives the advantage of being easy to apply and to interpret. More easy than, for example the likelihood ratio tests.

Moreover, it is quite fascinating to see how the tests use relevant residuals to test a specific hypothesis. This can clearly be seen in the Extended Growth Curve Model, where a particular residual is used in testing a particular hypothesis.

In order that the tests can be used in practice, we should be able to calculate the critical points. This requires the knowledge of the distributions. Unfortunately, the distributions are difficult to obtain. However, we show that the distributions under the corresponding null hypotheses are independent of the unknown covariance matrix Σ. Moreover, we suggest two reasonable approaches to get critical points for the tests.

The test statistic in (2.12), later given in the paper, is similar to the trace test suggested by Khatri (1966) and is usually known as the Lawley-Hotelling trace test for the Growth Curve Model. However, there is no explanation why Khatri made his suggestion and no one has tried to study the S_h and S_e matrices involved and see if they really are the variation matrices due to the hypothesis and the error as they are called by Khatri (1966) and many others, for example, Kariya (1978), Fujikoshi (1974) and Yanagihara (2001). That is one of the reasons why we should study the structure in the residuals which in turn helps us understand the structure of the tests.

We would like to note that, one arrives at the test given in (2.12) if one considers the Lawley-Hotelling trace test for the classical MANOVA model and use the transformation suggested by Potthoff & Roy (1964) with G=S. Moreover, according to Potthoff & Roy the choice of G=S seems to be appropriate for reasons we will explain later in the paper. It is also important to mention that the test given in (2.12) reduces to the Lawley-Hotelling trace test for the classical multivariate linear model. Because of these facts and due to the resemblance between the structure of the matrices involved in our tests and the S_h and S_e matrices given in Potthoff and Roy (1964), we may say that our tests are Lawley-Hotelling trace tests.

2. Hypothesis testing via residuals, the Growth Curve Model

Suppose that the Growth Curve Model given in Definition 1.1 has been fitted to data and we would like to know if the estimated model fits the data. In this section we formulate the hypothesis and propose suitable statistics for testing this hypothesis. We use the restricted maximum likelihood followed by estimated likelihood approaches to construct test, see Searle et al. (1992) for more details about the procedure. First we write the likelihood as a product of two terms and maximize the second part of the likelihood to get an estimator for the covariance matrix Σ. We use that information to get the estimated likelihood which is then maximized under H_o and $H_o \cup H_1$, where H_o and H_1 are the null and alternative hypotheses, respectively. We discuss why the tests appear natural and reasonable as well as are easy to apply in practice. We also try to interpret them in connection with the corresponding interpretation for the residuals given in Seid Hamid & von Rosen (2005a).

Let us suppose that we have fitted a Growth Curve Model and we would like to test if the estimated growth curve fits the data. The hypothesis can be formulated as follows,

$$\begin{aligned} H_o &: B = 0 \\ H_1 &: B \neq 0. \end{aligned} \tag{2.1}$$

Now consider the likelihood function for the Growth Curve Model which is given by

$$L = \alpha |\Sigma|^{-n/2} \exp\{-\frac{1}{2} tr\{\Sigma^{-1}(X - ABC)(X - ABC)'\}\}, \tag{2.2}$$

where $\alpha = (2\pi)^{-np/2}$. We can rewrite the above likelihood function as a product of two terms

$$\begin{aligned} L &= \alpha \exp\{-\frac{1}{2} tr\{\Sigma^{-1}(XC'(CC')^{-}C - ABC)(XC'(CC')^{-}C - ABC)'\}\} \\ &\quad \times |\Sigma|^{-n/2} \exp\{-\frac{1}{2} tr\{\Sigma^{-1}S\}\}, \end{aligned} \tag{2.3}$$

where $S = X(I - C'(CC')^{-}C)X'$. Note that the likelihood can be decomposed into two parts such that the resulting two components are themselves likelihood functions. This will give an unbiased estimator for the covariance matrix, however, we will obtain the same test statistic by using the decomposition given in (2.3).

Let us proceed by taking the second part of the likelihood which is given by

$$|\Sigma|^{-n/2} \exp\{-\frac{1}{2} tr\{\Sigma^{-1}S\}\}. \tag{2.4}$$

Maximize the above expression to get an estimator for the covariance matrix Σ which is given by

$$\hat{\Sigma} = \frac{1}{n}S. \tag{2.5}$$

We refer to Srivastava & Khatri (1979) to see how the estimator is obtained.

Once again, consider the likelihood function but this time use the estimator of the covariance matrix instead of the covariance matrix itself. The likelihood reduces to the following expression

$$EL = \alpha_1 |S|^{-\frac{n}{2}} e^{-\frac{n}{2} tr\{S^{-1}(XC'(CC')^{-}C - ABC)(XC'(CC')^{-}C - ABC)'\}}, \tag{2.6}$$

where $\alpha_1 = n^{n/2}(2\pi e)^{-np/2}$ and EL stands for estimated likelihood.

The next is to maximize the expression in (2.6) under the H_o and $H_o \cup H_1$. Under H_o, $B = 0$, the maximum of the estimated likelihood equals

$$\alpha_1 |S|^{-n/2} \exp\{-\frac{n}{2} tr\{S^{-1}XC'(CC')^{-}CX'\}\}. \tag{2.7}$$

Under $H_o \cup H_1$, the maximum can be obtained by replacing the observed mean structure ABC by its estimated maximum likelihood estimator which is given by

$$A\hat{B}C = A(A'S^{-1}A)^{-}A'S^{-1}XC'(CC')^{-}C.$$

The above expression is the estimated mean structure, which in fact is also the maximum likelihood estimator of the mean structure. For different ways of maximizing the likelihood function for the Growth Curve Model we refer to Srivastava & Khatri (1979) and Kollo & von Rosen (2005). Therefore, the maximum of (2.6) under the alternative becomes

$$\alpha_1 |S|^{-\frac{n}{2}} e^{-\frac{n}{2} tr\{S^{-1}(XC'(CC')^{-}C - A\hat{B}C)(XC'(CC')^{-}C - A\hat{B}C)'\}}. \tag{2.8}$$

Throughout the paper G^o is a matrix of full rank spanning the orthogonal complement to the column space of G. We can rewrite the residual R_{g3} given in (1.4) as follows:

$$R_{g3} = SA^o(A^{o\prime}SA^o)^{-}A^{o\prime}XC'(CC')^{-}C. \tag{2.9}$$

Moreover, it is possible to show that the residual R_{g3} can be written as the difference between the observed and estimated mean structures, i.e.,

$$R_{g3} = XC'(CC')^{-}C - A\hat{B}C.$$

Now define a test statistic by taking the ratio between (2.7) and (2.8), which can be written as

$$\frac{\exp\{-\frac{n}{2} tr\{S^{-1}XC'(CC')^{-}CX'\}\}}{\exp\{-\frac{n}{2} tr\{S^{-1}R_{g3}R_{g3}'\}\}}, \tag{2.10}$$

and the hypothesis is rejected for small values of the ratio. Note also that the ratio has values between zero and one. We can also use the test statistic which is obtained by taking the logarithm of the ratio. The test can be shown to be equivalent with

$$tr\{S^{-1}XC'(CC')^{-}CX'\} - tr\{S^{-1}R_{g3}R'_{g3}\}, \tag{2.11}$$

except that now we reject the hypothesis for large values of the above expression. The desired test which is given in the next proposition can then be reached by using the expression in (2.9) for R_{g3} and using the fact that $tr(AB) = tr(BA)$ for any two matrices A and B of proper sizes.

Proposition 2.1. *Suppose that the Growth Curve Model given in Definition 1.1 has been fitted to data and suppose that the hypothesis given in (2.1) is to be tested. A test statistic is given by*

$$\phi_1(x) = tr\{XC'(CC')^{-}CX'S^{-1}A(A'S^{-1}A)^{-}A'S^{-1}\}. \tag{2.12}$$

The hypothesis is rejected when $\phi_1(x)$ is large.

Observe that the test given in Proposition 2.1 is always greater or equal to zero. Moreover, it is possible to see from the equivalent expression in (2.10) that the numerator is a function of the observed mean structure, $XC'(CC')^{-}C$. On the other hand the denominator is a function of the residual, R_{g3} which is obtained by subtracting the estimated mean structure from the observed mean. That means the test compares the observed mean and the residuals. In other words, the test compares the observed and estimated means and rejects the hypothesis when they are "close" to each other, i.e., when the residuals are very "small". This characteristic of the test statistic we believe is very desirable and what makes the test natural, since it is a well known fact that comparing the observed and estimated values is the proper way to evaluate the estimated model.

In order that the test should be useful in practice one needs to know how large the value of the test statistic should be for the hypothesis to be rejected, i.e., we should be able to calculate the critical points. This requires the knowledge of the distribution of the test statistic which unfortunately is difficult to obtain. However, we suggest two alternative approaches which we believe are very important and useful as well as reasonable. The first one is to use a conditional test which is obtained by conditioning on the ancillary statistic, S. A detailed discussion of the test is presented in Seid Hamid & von Rosen (2005b). The second approach would be to approximate the density of the test based on moments.

Furthermore, under the null hypothesis the distribution of the test statistic is independent of Σ which is not obvious. This fact is shown in the following theorem. An important consequence of the theorem is that under the null hypothesis one can, without loss of generality, assume that $\Sigma = I$. As a result, the critical points are free of any unknown parameter. On the other hand, the distribution of the test under the alternative hypothesis depends on Σ. That means the power of the test depends on both Σ and B. However, one can use $\frac{1}{n}S$ instead of Σ to get an estimate of the power of the test which could be used as a measure of performance.

Theorem 2.2. *Consider the hypothesis in (2.1). Under the null hypothesis the distribution of the test given in (2.12) is independent of the unknown covariance matrix Σ.*

Proof. Let A^o be a matrix of full rank spanning the orthogonal complement to the space generated by the columns of A. We can write the test $\phi_1(x)$ as

$$\phi_1(x) = tr\{XC'(CC')^- CX'S^{-1}\} - tr\{XC'(CC')^- CX'A^o(A^{o\prime}SA^o)^{-1}A^{o\prime}\}.$$

The first term in the above expression is invariant under the transformation $\Sigma^{-\frac{1}{2}}X$. It is therefore possible to replace X by $\Sigma^{-\frac{1}{2}}X$ which shows that the distribution of first term in (2.13) is independent of Σ. For the second term, we can rewrite it as

$$tr\{C'(CC')^- CX'A^o(A^{o\prime}SA^o)^{-1}A^{o\prime}X\}.$$

Now, let us write $A^{o\prime}X$ as

$$(A^{o\prime}\Sigma A^o)^{\frac{1}{2}}(A^{o\prime}\Sigma A^o)^{-\frac{1}{2}}A^{o\prime}X.$$

Observe that we can rewrite $(A^{o\prime}\Sigma A^o)^{\frac{1}{2}}(A^{o\prime}SA^o)^{-1}(A^{o\prime}\Sigma A^o)^{\frac{1}{2}}$ as

$$((A^{o\prime}\Sigma A^o)^{-\frac{1}{2}}A^{o\prime}X(I - C'(CC')^- C)X'A^o(A^{o\prime}\Sigma A^o)^{-\frac{1}{2}})^{-1}.$$

Consequently, it remains to show that the distribution of $(A^{o\prime}\Sigma A^o)^{-\frac{1}{2}}A^{o\prime}X$ is independent of Σ. However, the expression is a linear function of a multivariate normal random variable, and as a result, it is enough to show that the mean and dispersion matrices are independent of Σ.

Under the null hypothesis $\boldsymbol{E}[X] = ABC = 0$ which implies

$$\boldsymbol{E}[(A^{o\prime}\Sigma A^o)^{-\frac{1}{2}}A^{o\prime}X] = 0.$$

Moreover,

$$\boldsymbol{D}[(A^{o\prime}\Sigma A^o)^{-\frac{1}{2}}A^{o\prime}X] = (A^{o\prime}\Sigma A^o)^{-\frac{1}{2}}A^{o\prime}\Sigma A^o(A^{o\prime}\Sigma A^o)^{-\frac{1}{2}} = I.$$

□

In the following theorem both the conditional and unconditional expectations for the test given in (2.12) are presented. One can see from the theorem that both types of expectations consist of two parts: one part which is independent of B, which in fact is the expected value of the test under the null hypothesis. The other part is an "increasing" function of B. That means the "more" B differs from 0, the more likely the hypothesis is to be rejected. In other words, it shows that the power of the test increases as B "increases".

Theorem 2.3. *Let $\phi_1(x)$ be as given in Proposition 2.1 and let $\mathbf{E}[\phi_1(x)|S]$ be the conditional expectation given S , then we have*

$$\begin{aligned} \boldsymbol{E}[\phi_1(x)] &= \beta\rho(C)\rho(A) + \beta tr\{ABC(ABC)'\Sigma^{-1}\}, \\ \boldsymbol{E}[\phi_1(x)|S] &= tr\{\rho(C)\Sigma S^{-1}A(A'S^{-1}A)^{-}A'S^{-1} + ABC(ABC)'S^{-1}\}, \end{aligned} \tag{2.13}$$

where $\beta = (n - \rho(C) - 1)^{-1}$ and $\rho(.)$ is the rank of a matrix.

Proof. The expression inside the trace function in (2.12) is the product of two independent terms. We can therefore write the expectation as

$$\boldsymbol{E}[\phi_1]tr\{\boldsymbol{E}[XC'(CC')^{-}CX']\boldsymbol{E}[S^{-1}A(A'S^{-1}A)^{-}A'S^{-1}]\}. \tag{2.14}$$

Observe that the first expectation on the right hand side of the above expression is the expectation of a noncentral Wishart random variable. Therefore,

$$\boldsymbol{E}[XC'(CC')^{-}CX'] = \rho(C)\Sigma + ABC(ABC)'.$$

For the second expectation, we write the expression in its canonical form to get

$$\boldsymbol{E}[S^{-1}A(A'S^{-1}A)^{-}A'S^{-1}] = \beta\Sigma^{-1}A(A'\Sigma^{-1}A)^{-}A'\Sigma^{-1}.$$

More on the canonical representation can be found in Seid Hamid (2001). The desired result can then be established after realizing that

$$tr\{A(A'\Sigma^{-1}A)^{-}A'\Sigma^{-1}\} = \rho(A)$$

and

$$A'\Sigma^{-1}A(A'\Sigma^{-1}A)^{-}A' = A'.$$

For the second statement, one can take out the second part of the expression which is a function of S. It remains to find the expected value of a noncentral Wishart variable which was given above. Moreover, it is important to note that

$$A'S^{-1}A(A'S^{-1}A)^{-}A' = A',$$

which completes the proof. □

Corollary 2.4. *Consider the hypothesis given in (2.1). Under the null hypothesis, the expectations in Theorem 2.3 reduce to*

$$\boldsymbol{E}[\phi_1(x)] = \beta\rho(C)\rho(A)$$
$$\boldsymbol{E}[\phi_1(x)|S] = \rho(C)tr\{S^{-1}A(A'S^{-1}A)^{-}A'S^{-1}\},$$

where β is as given in Theorem 2.3.

Observe that the test could also be applied in the classical multivariate model since the Growth Curve Model given in Definition 1.1 reduces to this model when $A = I$. The test statistic then reduces to

$$tr\{S^{-1}XC'(CC')^{-}CX'\}, \tag{2.15}$$

which is equivalent to the well known Lawley-Hotelling trace test or sometimes called the generalized Hotelling's test, which once again confirms that our test is a natural extension of the Lawely-Hotelling trace test to the Growth Curve Model.

Potthoff & Roy (1964) suggested that one can use the Lawley-Hotelling trace test for the classical multivariate linear model and use a transformation which is based on a constant G to get the trace test for the Growth Curve Model, i.e., they transformed X to $XG^{-1}P'(PG^{-1}P')^{-1}$. One suggestion they made was to choose G as an estimate of Σ but they also made it clear that in order for the test to perform better one should choose G such that it is independent of X. Now, if we look at the Lawley-Hotelling trace test for the classical multivariate linear model and use the transformation suggested by Potthoff & Roy (1964) with $G = S$, we will arrive at the test given in Proposition 2.1. Moreover, the choice of G seems appropriate since we can think of our problem as that of fitting a Growth Curve Model for the mean and S is independent of the mean.

As a final part of this section we would like to remind the reader that we have not assumed anything as to the linearity of the growth curves of the individuals involved in this model. This indicates that the methods utilized and hence the test defined can also be applied when the growth curves are polynomial of any degree. The only important thing that matters is that the degree of the polynomial should be the same for all individuals.

3. Hypothesis testing via the residuals, Extended Growth Curve Model

In this section we formulate practical hypotheses for the Extended Growth Curve Model given in Definition 1.2 and then construct tests using the restricted followed by estimated likelihood approaches. This model is more structured and we have more residuals than before. As a result, there are more hypotheses to formulate and test. We define four hypotheses which we believe are important and practical, and discuss them separately.

Before defining the possible hypotheses, we would like to note that we have used the same assumptions as in Seid Hamid & von Rosen (2005a). That is, we have, without loss of generality, assumed that there are two groups denoted by Group I and Group II and that the individuals in Groups I and II follow a linear and quadratic mean structure, respectively. Moreover, it is also supposed that there is a linear term in the growth curves of the individuals in Group II. See the paper by Seid Hamid & von Rosen (2005a) for notations and a detailed explanation of the situation.

Let us look at the following hypotheses, more details about the tests will be given later.

i) H_o: The estimated model fits the data

ii) H_o: The estimated linear growth curve fits the data for group I

iii) H_o: The estimated quadratic growth curve fits the data in group II

iv) H_o: The quadratic term in the growth curves is significant.

We write the likelihood function as a product of three independent terms. Depending on which hypothesis is considered, we maximize a certain part of the likelihood to get an estimator for the covariance matrix which then replaces the covariance matrix to give the estimated likelihood.

Consider the Extended Growth Curve Model given in Definition 1.2. We can rewrite it as

$$\begin{aligned} X &= A_1(B_{11} : B_{12}) \begin{pmatrix} C_{11} \\ C_{12} \end{pmatrix} + A_2 B_2 C_2 + \epsilon \\ &= A_1 B_{11} C_{11} + A_1 B_{12} C_{12} + A_2 B_2 C_2 + \epsilon, \end{aligned} \tag{3.1}$$

where $B_1 = (B_{11} : B_{12})$ and $C_1' = (C_{11}' : C_{12}')$. If we want to consider the two groups separately, the model reduces to

$$X_1 = A_1 B_{11} C_{11}^1 + \epsilon_1 \tag{3.2}$$

and

$$X_2 = A_1 B_{12} C_{12}^2 + A_2 B_2 C_{22} + \epsilon_2 \tag{3.3}$$

for Group I and Group II, respectively. Here X_1 and C_{11}^1 are matrices consisting of the first n_1 columns of X and C_{11}, respectively. The matrices X_2, C_{12}^2 and C_{22} consist of the last n_2 columns of X, C_{12} and C_2, respectively.

Observe that $C_{12} = C_2$. Moreover it is possible to show that

$$C_{11}'(C_{11}C_{11}')^- C_{11} = C_1'(C_1 C_1')^- C_1 - C_2'(C_2 C_2')^- C_2. \tag{3.4}$$

Suppose that we would like to test the first hypothesis given above against a two sided alternative. We can formulate it as follows

$$\begin{aligned} H_o &: B_1 = 0,\ B_2 = 0, \\ H_1 &: B_1 \neq 0,\ B_2 \neq 0. \end{aligned} \tag{3.5}$$

Now consider the likelihood function for the Extended Growth Model which is given by

$$L = \gamma |\Sigma|^{-\frac{n}{2}} e^{-\frac{1}{2} tr\{\Sigma^{-1}(X-(A_1B_1C_1+A_2B_2C_2))(X-(A_1B_1C_1+A_2B_2C_2))'\}}, \tag{3.6}$$

where $\gamma = (2\pi)^{-np/2}$. The likelihood can be rewritten as a product of three terms,

$$L = L_1 \ \times \ L_2 \ \times \ L_3, \tag{3.7}$$

where,

$$\begin{aligned} L_1 = \gamma \ \exp\{-\frac{1}{2} tr\{&\Sigma^{-1}(XC_2'(C_2C_2')^- C_2 - (A_1B_{12}C_{12} + A_2B_2C_2)) \\ &\times (XC_2'(C_2C_2')^- C_2 - (A_1B_{12}C_{12} + A_2B_2C_2))'\}\}, \\ L_2 = \exp\{-\frac{1}{2} tr\{&\Sigma^{-1}(X(C_1'(C_1C_1')^- C_1 - C_2'(C_2C_2')^- C_2) - (A_1B_{11}C_{11})) \\ &\times (X(C_1'(C_1C_1')^- C_1 - C_2'(C_2C_2')^- C_2) - (A_1B_{11}C_{11}))'\}\}, \\ L_3 = |\Sigma|^{-\frac{n}{2}} \exp\{-\frac{1}{2} tr\{&\Sigma^{-1}X(I - C_1'(C_1C_1')^- C_1)X'\}\}. \end{aligned}$$

Let us now consider the product of the last two expressions and use equation (3.4). Put $S_1 X(I - C_1'(C_1C_1')^- C_1)X'$ as in (1.12), then maximize the product to get an estimator for the covariance matrix Σ. The maximum estimated likelihood (maximum EL) estimator equals

$$\begin{aligned} n\hat{\Sigma} \ &= \ S_1 + (XC_{12}'(C_{12}C_{12}')^- C_{12} - A_1\hat{B}_{11}C_{11}) \\ &\quad \times (XC_{12}'(C_{12}C_{12}')^- C_{12} - A_1\hat{B}_{11}C_{11})', \end{aligned} \tag{3.8}$$

where $\hat{B}_{11}$ is the maximum EL estimator of B_{11}. In fact it is possible to show that it is also the maximum likelihood estimator. We refer to the paper by von Rosen (1989) for the maximum likelihood estimators and one can use his approach to maximize the estimated likelihood.

Let $\hat{B}_1$, $\hat{B}_{11}$ and $\hat{B}_{12}$ be the maximum EL estimates of B_1, B_{11} and B_{12}, respectively and let $\hat{B}_1 = (\hat{B}_{11} : \hat{B}_{12})$. Consequently, it is possible to show that the residual R_{e3} can be written as

$$R_{e3} = XC_{12}'(C_{12}C_{12}')^- C_{12} - A_1\hat{B}_{11}C_{11}. \tag{3.9}$$

Therefore the maximum estimated likelihood estimator of Σ equals

$$\hat{\Sigma} = \frac{1}{n} S_2,$$

where S_2 is as given in (1.13).

In the same way as we did for the Growth Curve Model, we replace Σ in (3.6) by its estimator, $\frac{1}{n}S_2$, to get the estimated likelihood, EL, and then maximize EL under H_o and $H_o \cup H_1$. The maximum of the EL under H_o and $H_o \cup H_1$ are respectively given by

$$\gamma_1 |S_2|^{-\frac{n}{2}} \exp\{-\frac{n}{2} tr\{S_2^{-1} X C_1'(C_1 C_1')^- C_1 X'\}\} \tag{3.10}$$

and

$$\begin{aligned} \gamma_1 |S_2|^{-\frac{n}{2}} \exp\{ &- \frac{n}{2} tr\{S_2^{-1}(XC_1'(C_1C_1')^- C_1 - (A_1\hat{B}_1C_1 + A_2\hat{B}_2C_2)) \\ &\times (XC_1'(C_1C_1')^- C_1 - (A_1\hat{B}_1C_1 + A_2\hat{B}_2C_2))'\}\}, \end{aligned} \tag{3.11}$$

where $\gamma_1 = n^{n/2}(2\pi e)^{-np/2}$. Note that we can rewrite R_{e3} and R_{e4} as follows:

$$R_{e3} = S_1 A_1^o (A_1^{o\prime} S_1 A_1^o)^- A_1^{o\prime} X (C_1'(C_1C_1')^- C_1 - C_2'(C_2C_2')^- C_2), \tag{3.12}$$

$$R_{e4} = S_2 A^o (A^{o\prime} S_2 A^o)^- A^o X C_2'(C_2C_2')^- C_2, \tag{3.13}$$

where A_1^o and A^o are matrices of full rank spanning the orthogonal complements of the column spaces of the matrices A_1 and $A = (A_1 : T_1 A_2)$, respectively.

Moreover, it is possible to show that R_{e34}, which denotes the sum of the residuals R_{e3} and R_{e4}, can be written as a difference between the observed and estimated means, i.e.,

$$R_{e34} = XC_1'(C_1C_1')^- C_1 - (A_1\hat{B}_1C_1 + A_2\hat{B}_2C_2).$$

Now as in the previous section a test statistic is defined by taking the ratio between (3.10) and (3.11). The statistics is given by

$$\frac{\exp\{-\frac{n}{2} tr\{S_2^{-1} X C_1'(C_1C_1')^- C_1 X'\}\}}{\exp\{-\frac{n}{2} tr\{S_2^{-1} R_{34} R_{34}'\}\}}, \tag{3.14}$$

where the hypothesis is rejected when the value of the ratio is small, i.e, close to zero. Note that the ratio has values between zero and one. One can also define an equivalent test by taking the logarithm of the test. This test statistic can be shown to be equivalent with

$$tr\{S_2^{-1} X C_1'(C_1C_1')^- C_1 X'\} - tr\{S_2^{-1} R_{34} R_{34}'\}, \tag{3.15}$$

We now reject the hypothesis for large values of (3.15).

Consider the first term in the above expression and write it as a sum of two terms as follows:

$$tr\{S_2^{-1}XC_1'(C_1C_1')^{-}C_1X'\} = tr\{S_2^{-1}XC_{12}'(C_{12}C_{12}')^{-}C_{12}X'\} + tr\{S_2^{-1}XC_2'(C_2C_2')^{-}C_2X'\}, \tag{3.16}$$

where C_{12} is as in (3.1). Similarly, use the expressions (3.12) and (3.13) for R_{e3} and R_{e4}, respectively, and write the second term in (3.15) as

$$tr\{S_2^{-1}R_{34}R_{34}'\} = tr\{XC_{12}'(C_{12}C_{12}')^{-}C_{12}X'S_2^{-1}A_1(A_1'S_2^{-1}A_1)^{-}A_1'S_2^{-1}\} + tr\{XC_2'(C_2C_2')^{-}C_2X'S_2^{-1}A(A'S_2^{-1}A)^{-}A'S_2^{-1}\}. \tag{3.17}$$

By subtracting (3.17) from (3.16) we get a test which will be given in the next proposition. Here it is important to note that the column spaces of $(A_1 : A_2)$ and $(A_1 : T_1A_2)$ are identical. This fact has been used in Seid Hamid (2001) when defining the residuals for the Extended Growth Curve Model.

Proposition 3.1. *Suppose that the Extended Growth Curve Model given in Definition 1.2 has been fitted to data and consider the hypothesis given in (3.5). A test statistic is given by*

$$\phi_2(x) = tr\{XC_{11}'(C_{11}C_{11}')^{-}C_{11}X'S_2^{-1}A_1(A_1'S_2^{-1}A_1)^{-}A_1'S_2^{-1}\} + tr\{XC_2'(C_2C_2')^{-}C_2X'S_2^{-1}A(A'S_2^{-1}A)^{-}A'S_2^{-1}\}, \tag{3.18}$$

where $A = (A_1 : T_1A_2)$. The hypothesis is rejected when the value of $\phi_2(x)$ is large.

The test given above is always greater or equal to zero. Moreover, it is possible to see from the expression in (3.14) that the numerator is a function of $XC_1'(C_1C_1')^{-}C_1$ which is the observed mean. On the other hand, in the denominator we have a function of R_{e34} which is the residual obtained by subtracting the estimated mean from the observed mean. This shows that the test compares the observed and estimated means and rejects the hypothesis when the difference between them is "small", in other words, when the residual R_{34} is "small".

The distribution of the test under the null hypothesis is independent of the unknown covariance matrix Σ. This fact is stated in the following theorem without a proof. However, similar proofs are given later and by combining these results one may verify the theorem.

Theorem 3.2. *Consider the hypothesis given in (3.5). Under the null hypothesis the distribution of the $\phi_2(x)$ is independent of the unknown covariance matrix Σ.*

The above theorem shows that under the null hypothesis we can, without loss of generality, assume that $\Sigma = I$. As a result, the critical points are free of any unknown parameter. However, as in $\phi_1(x)$, the power of the test depends of the unknown covariance matrix. One can, therefore, use an estimator of Σ to get the estimated power.

The conditional and unconditional expectations of $\phi_2(x)$ are given bellow. The theorem is stated without a proof. However, we show later that $\phi_2(x)$ is the sum of the two tests given in Propositions 3.5 and 3.9.

Theorem 3.3. *Let $\phi_2(x)$ be as given in Proposition 3.1. Let $A = (A_1 : A_2)$, $\mu_l = A_1 B_{11} C_{11}$ and $\mu_q = A_1 B_{12} C_{12} + A_2 B_2 C_2$. Then,*

$$\begin{aligned}\boldsymbol{E}[\phi_2(x)] &= \lambda\rho(C_{11})\rho(A_1) + \lambda tr\{\mu_l \mu_l' \Sigma^{-1}\} + \rho(C_2)\{\lambda\rho(A_1) \\ &\quad + \delta(\rho(A) - \rho(A_1))\} + tr\{\mu_q \mu_q' \{\lambda(I - \Sigma P_l) + \delta \Sigma P_l P_q P_l\}\}, \\ \boldsymbol{E}[\phi_2(x)|S_2] &= \{\rho(C_{11}) + \rho(C_2)\} tr\{S_1^{-1} A_1 (A_1' S_1^{-1} A_1)^{-} A_1' S_1^{-1}\} \\ &\quad + tr\{[\mu_l \mu_l' + \mu_q \mu_q'] S_2^{-1}\},\end{aligned}$$

where $\lambda = (n - \rho(C_1) - 1)^{-1}, \delta = (n - \rho(C_2) - 1)^{-1}$ and

$$\begin{aligned}P_l &= A_1^o (A_1^{o\prime} \Sigma A_1^o)^{-} A_1^{o\prime}, \\ P_q &= A_2 (A_2' P_l A_2)^{-} A_2'.\end{aligned}$$

Corollary 3.4. *Consider the hypothesis given in (3.5). Under the null hypothesis, the two expectations given in Theorem 3.3 reduce to*

$$\begin{aligned}\boldsymbol{E}[\phi_2(x)] &= \lambda\rho(C_{11})\rho(A_1) + \rho(C_2)\{\lambda\rho(A_1) + \delta(\rho(A) - \rho(A_1))\}, \\ \boldsymbol{E}[\phi_2(x)|S_2] &= \{\rho(C_{11}) + \rho(C_2)\} tr\{S_1^{-1} A_1 (A_1' S_1^{-1} A_1)^{-} A_1' S_1^{-1}\},\end{aligned}$$

where λ and δ are as given in Theorem 3.3.

Observe that we have replaced S_1^{-1} by S_2^{-1} to get the second expression in the above corollary. This was possible due to the fact that $A_1 S_1^{-1} = A_1 S_2^{-1}$.

Suppose that we are interested in just Group I and that we would like to test if the linear growth curve fits the data for Group I. We can approach this in two different ways. The first approach is to reduce the model to this group as in (3.2). As one can see the model reduces to the Growth Curve Model which we discussed in the previous section. We are going to discuss the second approach which is better than the first one in the sense that we use more information to estimate the covariance matrix which should give a better estimator which in turn gives a better test statistic.

Consider the hypothesis that the estimated linear growth curve fits the data, which can be formulated as

$$\begin{aligned} H_o &: B_{11} = 0, \\ H_1 &: B_{11} \neq 0. \end{aligned} \tag{3.19}$$

Let us start by looking at the likelihood function given in (3.6). Maximize the part of the likelihood which is denoted by L_3 to get an estimator for the covariance matrix, which equals nS_1. As usual, replace the covariance matrix in the likelihood by its estimator to get the estimated likelihood function and then maximize the estimated likelihood under H_o and $H_o \cup H_1$. The test is then constructed by taking the logarithm of the ratio multiplied by some constant, as in the previous cases. Similarly, it is possible to show that the resulting test is equivalent to the one given in Proposition 3.3 below. However, it is important to observe that one can replace S_1 by S_2 and vice versa due to the fact that $A_1'S_1^{-1} = A_1'S_2^{-1}$.

Proposition 3.5. *Suppose that the Extended Growth Curve Model has been fitted to data. Let the hypothesis to be tested be given by (3.19). A test statistic is given by*

$$\phi_3(x) = tr\{XC_{11}'(C_{11}C_{11}')^- C_{11}X'S_1^{-1}A_1(A_1'S_1^{-1}A_1)^- A_1'S_1^{-1}\}. \tag{3.20}$$

The hypothesis is rejected when the value of $\phi_3(x)$ is large.

Observe that the test given above can be written as a ratio of functions of the observed and estimated mean for the individuals in Group I. The test, therefore, compares these two values and rejects the hypothesis when the difference between them is small.

The test given in (3.20) is always greater or equal to zero. Moreover, it is shown in the following theorem that the distribution of the test under the null hypothesis is independent of the unknown covariance matrix Σ. However, as in the previous two tests, the distribution under the alternative will depend on Σ. We suggest that Σ could be replaced by its estimator $\frac{1}{n}S_1$ to get the estimated power.

Theorem 3.6. *Let the hypothesis to be tested be given in (3.19). The distribution of $\phi_3(x)$ under the null hypothesis is independent of the unknown covariance matrix Σ.*

Proof. We can rewrite $\phi_3(x)$ as

$$\begin{aligned} \phi_3(x) = &\; tr\{XC_{11}'(C_{11}C_{11}')^- C_{11}X'S_1^{-1}\} \\ &- tr\{XC_{11}'(C_{11}C_{11}')^- C_{11}X'A_1^o(A_1^{o\prime}S_1A_1^o)^- A_1^{o\prime}\}. \end{aligned}$$

The first term in the above expression is invariant under the transformation $\Sigma^{-\frac{1}{2}}X$. One can therefore replace X in the expression with $\Sigma^{-\frac{1}{2}}X$ which shows that the distribution is independent of Σ. The second term can be written as

$$tr\{C_{11}'(C_{11}C_{11}')^{-}C_{11}X'A_1^o(A_1^{o\prime}S_1A_1^o)^{-}A_1^{o\prime}XC_{11}'(C_{11}C_{11}')^{-}C_{11}\}.$$

It remains to show that the distribution of $A_1^{o\prime}XC_{11}'(C_{11}C_{11}')^{-}C_{11}$ is independent of Σ. This can be shown by using calculations similar to those used earlier when proving Theorem 2.2. However, it is important to note that, under the null hypothesis

$$\boldsymbol{E}[A_1^{o\prime}XC_{11}'(C_{11}C_{11}')^{-}C_{11}] = 0.$$

□

In the following theorem, both conditional and unconditional expectations are given. Both types of expectations consist of two parts. One is independent of the parameter B_{11}. The second one is an "increasing" function of the parameter. As a result, the more B_{11} differs from 0, the more likely the hypothesis is to be rejected. This indicates that the power of the test "increases" with B_{11}.

Theorem 3.7. *Let $\phi_3(x)$ be as given in Proposition 3.5 and let λ and μ_l be as given in Theorem 3.3. Then,*

$$\boldsymbol{E}[\phi_3(x)] = \lambda\rho(C_{11})\rho(A_1) + \lambda tr\{\mu_l\mu_l'\Sigma^{-1}\},$$
$$\boldsymbol{E}[\phi_3(x)|S_1] = tr\{\rho(C_{11})\Sigma S_1^{-1}A_1(A_1'S_1^{-1}A_1)^{-}A_1'S_1^{-1} + \mu_l\mu_l'S_1^{-1}\},$$

Proof. The proof is similar with that of Theorem 2.3. The only difference is that we have C_{11} and S_1 instead of C and S. It is important to note that

$$\boldsymbol{E}[XC_{11}'(C_{11}C_{11}')^{-}C_{11}] = A_1B_{11}C_{11}.$$

□

Corollary 3.8. *Consider the hypothesis given in (3.19) and let λ be as given in Theorem 3.3. Under the null hypothesis, the two expectations given in Theorem 3.7 reduce to*

$$\boldsymbol{E}[\phi_3(x)] = \lambda\rho(C_{11})\rho(A_1),$$
$$\boldsymbol{E}[\phi_3(x)|S_1] = \rho(C_{11})tr\{S_1^{-1}A_1(A_1'S_1^{-1}A_1)^{-}A_1'S_1^{-1}\},$$

If we are interested in Group II and want to test the hypothesis that the estimated quadratic growth curve fits the data in Group II. We can also do it in two different ways. The first one is to use the model given in (3.3) and reduce the problem to the Growth Curve Model case. As in the previous case, the second alternative that we shall discuss is better for the same reasons mentioned earlier. The approach is to look at the likelihood function given in (3.6) and maximize $|\Sigma|^{-\frac{n}{2}} L_1 \times L_2$ to get an estimator for Σ and update the likelihood based on the estimator and proceed as before. One can show that the test constructed is identical to the one given in the following proposition.

Proposition 3.9. *Suppose the Extended Growth Curve Model has been fitted to data. Consider the following hypothesis*

$$H_o : B_{11} = 0,\ B_2 = 0,$$
$$H_1 : B_{11} \neq 0\ ,\ B_2 \neq 0.$$

Let $A = (A_1 : T_1 A_2)$, then a test statistic is given by

$$\phi_4(x) = tr\{XC_2'(C_2C_2')^- C_2 X' S_2^{-1} A (A' S_2^{-1} A)^- A' S_2^{-1}\}, \tag{3.21}$$

and the hypothesis is rejected when the value of $\phi_4(x)$ is large.

The above test, like the previous tests, is greater or equal to zero. Moreover, it is shown in the following theorem that its distribution under the null hypothesis will not depend on Σ. This shows that, under the null hypothesis, we can assume that $\Sigma = I$. As a result, the critical points will not depend on Σ. However, the power of the test depends on Σ. One can use $\frac{1}{n}S_2$ as an estimator of Σ to get the estimated power.

Theorem 3.10. *Consider the hypothesis given in Proposition 3.9. The distribution of $\phi_4(x)$ under the null hypothesis does not depend on the unknown covariance matrix Σ*

Proof. Consider the expression in (3.21) and write it as

$$\begin{aligned}\phi_4(x) &= tr\{XC_2'(C_2C_2')^- C_2 X' S_2^{-1} A_1 (A_1' S_2^{-1} A_1)^- A_1' S_2^{-1}\} \\ &\quad + tr\{XC_2'(C_2C_2')^- C_2 X' S_2^{-1} T_1 A_2 (A_2' T_1' S_2^{-1} T_1 A_2)^- A_2' T_1 S_2^{-1}.\}\end{aligned}$$

Let us denote the two terms in the above expression by **I** and **II**, respectively. For the term denoted by **I**, we can replace S_2^{-1} by S_1^{-1}. The expression can then be rewritten to give a similar expression as in (2.13). One can then use similar arguments to show that its distribution is independent of Σ. On the other hand, the term denoted by **II** can be written as the difference between two terms as follows:

$$\begin{aligned}\mathbf{II} \ &= \ tr\{XC_2'(C_2C_2')^- C_2 X' G_1 (G_1' W_2 G_1)^{-1} G_1'\} \\ &\quad - tr\{XC_2'(C_2C_2')^- C_2 X' G_2 (G_2' W_2 G_2)^{-1} G_2'\},\end{aligned} \tag{3.22}$$

where

$$G_{r+1} = G_r (G_r' A_{r+1})^o,\ \ G_0 = I,$$
$$W_{r+1} = X(I - C_r'(C_r C_r')^- C_r) X'.$$

Such approaches have been discussed in a more general form in von Rosen (1990). For special cases like the above one, we refer to Seid Hamid (2001).

The two terms in (3.22) can be rewritten as

$$tr\{C_2'(C_2C_2')^-C_2X'G_1(G_1'W_2G_1)^{-1}G_1'X\}$$

and

$$tr\{C_2'(C_2C_2')^-C_2X'G_2(G_2'W_2G_2)^{-1}G_2'X\}.$$

We want to show that the distributions of the above two expressions under the null hypothesis are independent of Σ. It is equivalent if we show that the distributions of $G_1'X$ and $G_2'X$ under the null hypothesis are independent of Σ.

Now write $G_1'X$ as

$$(G_1\Sigma G_1')^{\frac{1}{2}}(G_1\Sigma G_1')^{-\frac{1}{2}}G_1'X',$$

it remains to show that the distribution of $(G_1\Sigma G_1')^{-\frac{1}{2}}G_1'X'$, which is a linear function of a multivariate normal random variable, is independent of Σ. Because of normality, it is enough to show that the mean and dispersion matrices are independent of Σ.

Under the null hypothesis, $B_2 = 0$. Moreover, $G_1'A_1 = A_1^{o\prime}A_1 = 0$. Consequently, we have

$$\boldsymbol{E}[(G_1\Sigma G_1')^{-\frac{1}{2}}G_1'X] = (G_1\Sigma G_1')^{-\frac{1}{2}}G_1'(A_1B_1C_1 + A_2B_2C_2) = 0.$$

Furthermore,

$$\boldsymbol{D}[G_1\Sigma G_1')^{-\frac{1}{2}}G_1'X] = (G_1\Sigma G_1')^{-\frac{1}{2}}G_1'\Sigma G_1(G_1\Sigma G_1')^{-\frac{1}{2}} = I.$$

Finally, similar calculations can be used to show that the distribution of $G_2'X$ is independent of Σ. □

As mentioned earlier for the previous tests, the expectations given in Theorem 3.11 have two components where one part is an "increasing" function of the parameters of interest. Moreover, one can decompose each component into two different parts, a linear and a quadratic part. This is because we have assumed that the quadratic growth curves for the individuals in Group II consist of a linear term.

Theorem 3.11. *Let $\phi_4(x)$ and A be as in Proposition 3.9. Let λ, δ, P_l, P_q and μ_q be as given in Theorem 3.3. Then*

$$\begin{aligned}\boldsymbol{E}[\phi_4(x)] &= \rho(C_2)\{\lambda\rho(A_1) + \delta(\rho(A) - \rho(A_1))\} + tr\{\mu_q\mu_q'\{\lambda(I - \Sigma P_l)\\ &\quad + \delta\Sigma P_lP_qP_l\}\},\\ \boldsymbol{E}[\phi_4(x)|S_2] &= tr\{\rho(C_2)\Sigma S_2^{-1}A(A'S_2^{-1}A)^-A'S_2^{-1} + \mu_q\mu_q'S_2^{-1}\},\end{aligned}$$

Proof. We can write the expectation in the first statement as a product of two terms

$$\boldsymbol{E}[\phi_4(x)] = tr\{\boldsymbol{E}[XC_2'(C_2C_2')^- C_2X']\boldsymbol{E}[S_2^{-1}A(A'S_2^{-1}A)^- A'S_2^{-1}]\}, \tag{3.23}$$

The matrix $XC_2'(C_2C_2')^- C_2X'$ has a non-central Wishart distribution , and therefore, we have the following

$$\boldsymbol{E}[XC_2'(C_2C_2')^- C_2X'] = \rho(C_2)\Sigma + \mu_q\mu_q'. \tag{3.24}$$

The second expectation on the right hand side of (3.23) can be written as

$$\boldsymbol{E}[S_2^{-1}A_1(A_1'S_2^{-1}A_1)^- A_1'S_2^{-1}] + \boldsymbol{E}[S_2^{-1}T_1A_2(A_2'T_1'S_2^{-1}T_1A_2)^- A_2'T_1'S_2^{-1}]. \tag{3.25}$$

Consider the first term on the right hand side. We can replace S_2 by S_1 as mentioned earlier, see Seid Hamid (2001). Using similar methods as before, one can then show that

$$\boldsymbol{E}[S_2^{-1}A_1(A_1'S_2^{-1}A_1)^- A_1'S_2^{-1}] = \lambda\Sigma^{-1}A_1(A_1'\Sigma^{-1}A_1)A_1'\Sigma^{-1}. \tag{3.26}$$

For the second term in (3.25), we can use the same method as in (3.22) and write it as

$$\boldsymbol{E}[G_1(G_1'W_2G_1)^{-1}G_1'] - \boldsymbol{E}[G_2(G_2'W_2G_2)^{-1}G_2'], \tag{3.27}$$

The two expectations in (3.27) are relatively easy to compute since W_2 is a Wishart random variable and the expression in (3.27) equals

$$\delta G_1(G_1'\Sigma G_1)^{-1}G_1'A_2(A_2'G_1(G_1'\Sigma G_1)^{-1}G_1'A_2)^- A_2'G_1(G_1'\Sigma G_1)^{-1}G_1'.$$

Now replace G_1 in the above expression by A_1^o which gives

$$\boldsymbol{E}[S_2^{-1}T_1A_2(A_2'T_1'S_2^{-1}T_1A_2)^- A_2'T_1S_2^{-1}] = \delta P_lP_qP_l. \tag{3.28}$$

The desired result can then be reached by combining the results in (3.24), (3.26) and (3.28), but first observe that the column spaces of the matrices $(A_1 : A_2)$ and $(A_1 : T_1A_2)$ are identical, and that $tr\{\Sigma P_lP_qP_l\} = \rho(A) - \rho(A_1)$.

For the conditional expectation, one can easily show that the statement holds after taking out the second part of the expression which is a function of S_2. □

Corollary 3.12. *Let the hypothesis to be tested be as given in Proposition 3.9. Under the null hypothesis the two expectations in Theorem 3.11 reduce to*

$$\boldsymbol{E}[\phi_4(x)] = \rho(C_2)\{\lambda\rho(A_1) + \delta(\rho(A) - \rho(A_1))\},$$
$$\boldsymbol{E}[\phi_4(x)|S_2] = \rho(C_2)tr\{S_2^{-1}A(A'S_2^{-1}A)^- A'S_2^{-1}\},$$

The test given in Proposition 3.9 compares the observed and estimated means for the individuals in Group II and rejects the hypothesis when the difference between them is small. That is when the residual, R_{e4} is small. Observe also that the tests $\phi_3(x)$ and $\phi_4(x)$, respectively, are equivalent to the first and second terms of the test given in (3.18). This shows that $\phi_2(x)$, as expected, combines both information for testing the hypothesis that the overall estimated model fits the data. Moreover, one can combine the proofs of Theorems 3.6 and 3.11 to prove Theorem 3.3.

Finally, suppose that we want to check if the quadratic term in the growth curves of the individuals in the second group is significant, i.e., if we should keep the quadratic term or not. We can formulate the hypotheses as

$$\begin{aligned} H_o &: B_2 = 0 \\ H_1 &: B_2 \neq 0. \end{aligned} \tag{3.29}$$

Let us, for the last time, consider the likelihood function given in (3.6) and maximize the product, $L_2 \times L_3$, to get the estimator S_2 of $n\Sigma$. Update the likelihood and proceed as usual by taking the maximum of the estimated likelihood under H_o and $H_o \cup H_1$. The test is then defined by taking the logarithm of the ratio between them where the hypothesis rejected for large values. The test can be shown to be equivalent with the one given in Proposition 3.13 by using $tr(AB) = tr(BA)$ several times and the fact that

$$\begin{aligned} I - A_1(A_1' S_2^{-1} A_1)^- A_1' S_2^{-1} T_1 A_2 (A_2' T_1' S_2^{-1} A_2 T_1)^- A_2' T_1' S_2^{-1} \\ + S_2 A^o (A^{o\prime} S_2 A^o) A^{o\prime}, \end{aligned} \tag{3.30}$$

where A^o and T_1 are as mentioned before.

Note that the two terms on the right hand side of (3.30) are orthogonal to each other. We refer to Seid Hamid (2001) for further explanations of decompositions of the spaces involved in defining the residuals.

Proposition 3.13. *Suppose that the Extended Growth Curve Model has been fitted to data and consider the hypothesis given in (3.29). A test statistic is given by*

$$\phi_5(X) = tr\{XC_2'(C_2C_2')^- C_2 X' S_2^{-1} T_1 A_2 (A_2' T_1' S_2^{-1} A_2 T_1)^- A_2' T_1' S_2^{-1}\}. \tag{3.31}$$

The hypothesis is rejected when the value of $\phi_5(x)$ is large.

The above test is always greater or equal to zero. Moreover, the ratio of the EL under H_o and $H_o \cup H_1$ can be written as

$$\frac{\exp\{-\frac{n}{2} tr\{S_2^{-1} R_{e4}^l {R_{e4}^l}'\}\}}{\exp\{-\frac{n}{2} tr\{S_2^{-1} R_{e4} R_{e4}'\}\}}, \tag{3.32}$$

where,

$$R^l_{e4} = (I - A_1(A_1'S_2^{-1}A_1)^- A_1'S_2^{-1})XC_2'(C_2C_2')^- C_2, \tag{3.33}$$

and the hypothesis is rejected when the value of the ratio in (3.32) is small. If we look at R^l_{e4} carefully, we understand that it is the residual R_{e4} obtained if we fitted a linear growth curve instead of a quadratic one for individuals in Group II. As expected, the test compares this value and the value of R_{e4} given in (1.9). That means the test checks if the contribution of the quadratic term is significant by comparing these two values. If values of R_{e4} are small compared to the values of R^l_{e4}, the value of the test statistic will then be large which leads to the rejection of the hypothesis which means that there is a need to keep the quadratic term since its contribution is significant.

The distribution of $\phi_5(x)$ under the null hypothesis is independent of Σ. This is given in the following theorem without a proof.

Theorem 3.14. *Suppose the hypothesis to be tested is as given in (3.29). Under the null hypothesis, the distribution of the $\phi_5(x)$ is independent of Σ.*

The conditional and unconditional expectations of the $\phi_5(x)$ are given in the following theorem which is stated without a proof because similar calculations have been made earlier in this section. As one can see from the theorem, the expectations have two parts. The first part is independent of B_2, whereas, the second one is an "increasing" function of the parameter. Moreover, it is important to note that the terms involved are the quadratic components of the expectations given in Theorem 3.11 which is quite natural, because, here we are only interested in the quadratic term.

Theorem 3.15. *Suppose $\phi_5(x)$ is as given in (3.31) and let δ be as given in Theorem 3.3. Then*

$$\begin{aligned}
\boldsymbol{E}[\phi_5] &= \rho(C_2)\delta\rho(A_2) + \delta tr\{(A_2B_2C_2)(A_2B_2C_2)'\Sigma^{-1}\},\\
\boldsymbol{E}[\phi_5|S_2] &= tr\{\rho(C_2)\Sigma S_2^{-1}T_1A_2(A_2'T_1'S_2^{-1}A_2T_1)^- A_2'T_1'S_2^{-1}\\
&\quad + (A_2B_2C_2)(A_2B_2C_2)'S_2^{-1}\}.
\end{aligned}$$

Corollary 3.16. *Consider the hypothesis in (3.29). Under the null hypothesis the two expectations in Theorem 3.15 reduce to*

$$\begin{aligned}
\boldsymbol{E}[\phi_5] &= \delta\rho(C_2)\rho(A_2),\\
\boldsymbol{E}[\phi_5|S_2] &= \rho(C_2)tr\{S_2^{-1}T_1A_2(A_2'T_1'S_2^{-1}A_2T_1)^- A_2'T_1'S_2^{-1}\}.
\end{aligned}$$

We would like to note that the individuals in Groups I and II do not have to follow a linear and a quadratic mean structure. The methods utilized can easily be modified if the mean structures follow any other polynomials. The only necessary assumption is the nested subspace condition mentioned in

Definition 1.2 which was used in defining the residuals. We refer the reader to Seid Hamid & von Rosen (2005a) to see where they have used this assumption when defining the residuals. Moreover, one can use similar approaches to extend the results in this section to the general Extended Growth Curve Model except that it may not be possible to get nice expressions for the residuals as well as the tests.

4. Concluding remarks

Tests for checking the adequacy of the model for the Growth and Extended Growth Curve models have been proposed. The tests were constructed using the restricted followed by estimated likelihood approaches. The covariance matrix Σ was estimated from one part of the likelihood function, then Σ was replaced by its estimator to get the estimated likelihood. The tests were then defined by taking the logarithm of the ratio between the maximum of the estimated likelihood under the null and alternative hypotheses. We have presented a summary of the tests together with their decision rules in Table 1.

The tests turned out to be functions of the residuals defined utilizing the bilinear structures in the corresponding models. This enabled us to study the structure behind the tests and interpret them in accordance with the interpretation given by Seid Hamid & von Rosen (2005a). Moreover, we have discovered resemblance between our tests and the Lawley-Hotelling's trace test for the classical multivariate linear model.

In practice, we need to find the critical points for all the tests constructed in this paper so that we could actually be able to use them. This requires the knowledge of the distribution of the tests which, unfortunately, are difficult to obtain. However, one can use several approaches to calculate the critical points. One is to condition on an ancillary statistic, this approach will be considered in our forthcoming paper. The distribution for the conditional tests is relatively easy to deal with. Moreover, conditioning reduces the inference to the situation at hand. Furthermore, we don't lose any information about the parameters of interest since the statistics we condition on is ancillary. Another alternative approach is to approximate the density for the tests using the first two moments. Such approach has been utilized by von Rosen (1995) when he approximated the densities for the three residuals he defined for the Growth Curve Model. In this paper we have given both the conditional and unconditional expectations for the tests. They all have two components: one part which is independent of the parameters of interest and a second component which in some sense is an "increasing" function of the parameter of interest.

We have also shown that, under the null hypotheses, the distribution of the five tests proposed in this paper are all independent of the unknown covariance matrix Σ. As a result, the critical points are free of any unknown parameter. Unfortunately, we can not say the same thing about the distributions under the alternative hypotheses. Consequently, the powers of the tests depend on both the parameters of interest and Σ. We have suggested that an appropriate estimator for Σ can be used to get the estimated powers which could be used as a measures of performance of the tests.

As a final part of the paper we would like to note that the hypothesis given in (2.1) can be formulated in its general form, i.e.

$$H_o : FBG = 0,$$
$$H_1 : FBG \neq 0,$$

where F and G are any two matrices. This kind of general formulation can for example be used if one is interested in comparing two growth curves which could be done by choosing suitable elements for the matrices F and G. Similarly, the hypotheses given for the Extended Growth Curve Model can also be formulated in a more general form.

Table 1. *Hypotheses in two GMANOVA Models and corresponding tests with their decision rules.*

Model	Hypothesis	Test	Decision rule
Growth Curve Model $X = ABC + \epsilon$	$H_o : B = 0$ $H_1 : B \neq 0$	$\phi_1(x)$	reject when $\phi_1(x)$ is large
Extended Growth Curve Model	$H_0 : B_1 = 0,\ B_2 = 0$ $H_1 : B_1 \neq 0,\ B_2 \neq 0$	$\phi_2(x)$	reject when $\phi_2(x)$ is large
	$H_o : B_{11} = 0$ $H_1 : B_{11} \neq 0$	$\phi_3(x)$	reject when $\phi_3(x)$ is large
$X = A_1B_1C_1 + A_2B_2C_2 + \epsilon$ $= A_1B_{11}C_{11} + A_1B_{12}C_{12}$	$H_0 : B_{11} = 0,\ B_2 = 0$ $H_1 : B_{11} \neq 0,\ B_2 \neq 0$	$\phi_4(x)$	reject when $\phi_4(x)$ is large
$+A_2B_2C_2 + \epsilon$	$H_o : B_2 = 0$ $H_1 : B_2 \neq 0$	$\phi_5(x)$	reject when $\phi_5(x)$ is large

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