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Research Report
Centre of Biostochastics

Swedish University of Agricultural Sciences

Report 2007:4
ISSN 1651-8543

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Abstract

Inference for the coefficient of variation in normally distributed populations is considered. An explicit estimator of a coefficient of variation that is shared by several populations is proposed. Methods for making confidence intervals and statistical tests, based on McKay's approximation for the coefficient of variation, are provided. Exact expressions for the first two moments of McKay's approximation are given. An approximate F-test for equality of a coefficient of variation that is shared by several populations and a coefficient of variation that is shared by several other populations is introduced. A simulation study of the performance of this test is performed.

Keywords: Relative error; McKay's approximation; confidence interval; likelihood ratio test.

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1 Introduction

The coefficient of variation c in a single sample with observations $y_1, y_2, \ldots, y_n$ is defined as $c = s/m$, where m and s are

$$m = \frac{1}{n}\sum_{j=1}^{n} y_j \qquad \text{and} \qquad s = \sqrt{\frac{1}{n-1}\sum_{j=1}^{n}(y_j - m)^2}, \tag{1}$$

respectively. In this paper we consider independently and normally distributed observations with expected value $\mu > 0$, variance σ^2 and population coefficient of variation $\gamma = \sigma/\mu$.

We discuss three problems:

(i) Estimation of a coefficient of variation γ that is shared by k populations;

(ii) Confidence interval and test for a coefficient of variation γ that is shared by k populations;

(iii) Test for equality of a coefficient of variation γ_1 that is shared by k_1 populations and a coefficient of variation γ_2 that is shared by k_2 populations.

Given k estimates $c_1, \ldots, c_k$ of a population coefficient of variation γ that is shared by k populations, a method is needed for pooling the estimates into one estimate. Explicitly we want to know if the single estimates shall be weighted by the number of observations n_i, by the degrees of freedom $n_i - 1$ or by some other function of the sample size.

Zeigler (1973) compared several estimators of a coefficient of variation that is shared by k populations, but considered only the case of equally large sample sizes, and did not discuss hypothesis tests and confidence intervals. Tian (2005) studied the problem of making inference about a common γ based on k samples and suggested a repeated sampling method for calculating a generalized probability value as defined by Tsui and Weerahandi (1989). A drawback with resampling methods is that they do not give the same result whenever applied. There could be a need for a method based on explicit expressions. Verrill and Johnson (2007) proposed a likelihood ratio based confidence interval for a coefficient of variation that is shared by k populations. However, the likelihood ratio test is computationally inconvenient when there are many populations, and we shall in this paper (Section 4.2) see that it has too large probability of type I error when the sample sizes are small.

A confidence interval and a test for a coefficient of variation that is shared by k populations could instead be based on the transformation for the coefficient of variation developed by McKay (1932). This transformation gives an approximately χ^2 distributed random variable when $\gamma < 1/3$, as confirmed by Fieller (1932), Pearson (1932), Iglewicz and Myers (1970), and Umphrey (1983). Forkman and Verrill (2007) showed that McKay's χ^2 approximation is asymptotically normal with mean $n-1$ and variance slightly smaller than $2(n-1)$. Vangel (1996) showed, by Taylor series expansion, that the error in McKay's approximation is small when the population coefficient of variation is small. In this paper we derive exact expressions for the first two moments of McKay's approximation.

A test is also introduced, based on McKay's transformation, for the equality of a coefficient of variation that is shared by k_1 populations and a coefficient of variation that is shared by k_2 populations. Many tests have been proposed for the special case $k_1 = k_2 = 1$: the likelihood ratio test (Lohrding, 1975; Bennett, 1977; Doornbos and Dijkstra, 1983), the Wald test and the score test (Gupta and Ma, 1996), Bennett's test (Bennett, 1976; Shafer and Sullivan, 1986), and Miller's test (Miller, 1991a; Feltz and Miller, 1996; Miller and Feltz, 1997).

The three problems listed above are considered in Section 2 – 4, respectively. In Section 4.2 we make a small Monte Carlo study of the performance of the new test for equality of coefficients of variation compared with the likelihood ratio test, Bennett's test and Miller's test.

2 Estimation of a coefficient of variation that is shared by k populations

Let us consider samples from k normally distributed populations with a common population coefficient of variation γ, and define the sample coefficients of variation as in Definition 1.

Definition 1 *Let $y_{ij} = \mu_i + e_{ij}$, where e_{ij} are independently distributed $N(0, \sigma_i^2)$, $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n_i$, with positive expected values μ_i and a positive population coefficient of variation $\gamma = \sigma_i/\mu_i$ that is shared by the k populations. The coefficient of variation c_i of sample i, $i = 1, 2, \ldots, k$, is defined as $c_i = s_i/m_i$, where m_i and s_i are*

$$m_i = \frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij} \qquad \textit{and} \qquad s_i = \sqrt{\frac{1}{n_i - 1}\sum_{j=1}^{n_i}(y_{ij} - m_i)^2}, \tag{2}$$

respectively.

We shall, throughout this paper, assume that the population coefficient of variation γ is smaller than $1/3$. By this assumption the expected value is larger than 3 standard deviations, and the probability of negative observations is, for small sample sizes, negligible.

The joint probability density function of the observations $\{y_{ij}\}$ can be written as

$$\prod_{i=1}^{k}(2\pi)^{-n_i/2}\exp\left(\frac{1}{\mu_i\gamma^2}\sum_{j=1}^{n_i}y_{ij}-\frac{1}{2\mu_i^2\gamma^2}\sum_{j=1}^{n_i}y_{ij}^2-\frac{n_i}{2\gamma^2}+n_i\log\mu_i\gamma\right). \qquad (3)$$

Thus, by the factorization theorem, the $2k$ dimensional statistic

$$S=\left\{\sum_{j=1}^{n_i}y_{ij}\,,\sum_{j=1}^{n_i}y_{ij}^2\right\}_{i=1}^{k}$$

is sufficient for $\eta = \{1/(\mu_i\gamma^2)\,,1/(\mu_i^2\gamma^2)\}$, and since there is a one-to-one correspondence between η and $\beta = (\gamma,\mu_1,\mu_2,\ldots,\mu_k)$, S is also sufficient for β. By writing (3) as

$$\exp\left(-\frac{1}{2\gamma^2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{(y_{ij}-\mu_i)^2}{\mu_i^2}+\sum_{i=1}^{k}n_i\log\mu_i\gamma-\sum_{i=1}^{k}\frac{n_i}{2}\log(2\pi)\right),$$

we see that

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{(y_{ij}-\mu_i)^2}{\mu_i^2}$$

is complete and sufficient for γ^2, when μ_i, $i=1,2,\ldots,k$, are known. When μ_i, $i=1,2,\ldots,k$, are unknown, μ_i could be estimated by m_i. Thus consider

$$U=\frac{1}{\sum_{i=1}^{k}(n_i-1)}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{(y_{ij}-m_i)^2}{m_i^2}=\frac{\sum_{i=1}^{k}(n_i-1)\,c_i^2}{\sum_{i=1}^{k}(n_i-1)}\;, \qquad (4)$$

with notation according to Definition 1, as an estimator of γ^2.

Theorem 1 *Let γ be a coefficient of variation that is shared by k populations, as defined in Definition 1. Let $v=\sum_{i=1}^{k}(n_i-1)$ and $U_v=U$ as defined by (4). Assume that $(n_i-1)/v\rightarrow\lambda_i>0$ as $v\rightarrow\infty$. Then*

$$\sqrt{v}\;(U_v-\gamma^2)\overset{d}{\longrightarrow}N(0\,,2\gamma^4+4\gamma^6),\quad \textit{as } v\rightarrow\infty.$$

Proof. With notations according to Definition 1

$$\sqrt{n_i}\ (m_i - \mu_i, s_i^2 - \sigma_i^2) \xrightarrow{d} \mathrm{N}(\mathbf{0}\,, \mathbf{V}), \quad i = 1, 2, \ldots, k,$$

where

$$\mathbf{V} = \begin{pmatrix} \sigma_i^2 & 0 \\ 0 & 2\sigma_i^4 \end{pmatrix}.$$

Referring to Serfling (1980, p.124)

$$\sqrt{n_i}\ (c_i^2 - \gamma^2) \xrightarrow{d} \mathrm{N}(0\,, \mathbf{g}'\mathbf{V}\mathbf{g}), \quad i = 1, 2, \ldots, k,$$

where, evaluated at $\{m_i, s_i^2\} = \{\mu_i, \sigma_i^2\}$,

$$\mathbf{g}' = \left(\frac{\partial c_i^2}{\partial m_i}, \frac{\partial c_i^2}{\partial s_i^2} \right) = \left(\frac{-2\sigma_i^2}{\mu_i^3}, \frac{1}{\mu_i^2} \right).$$

Thus

$$\sqrt{n_i}\ (c_i^2 - \gamma^2) \xrightarrow{d} \mathrm{N}(0, 2\gamma^4 + 4\gamma^6), \quad i = 1, 2, \ldots, k,$$

and, since $\sum_{i=1}^{k} \lambda_i = 1$,

$$\sqrt{v}\ (U - \gamma^2) = \sum_{i=1}^{k} \sqrt{\frac{n_i - 1}{v}}\ \sqrt{n_i - 1}\ (c_i^2 - \gamma^2) \xrightarrow{d} \mathrm{N}(0\,, 2\gamma^4 + 4\gamma^6). \quad \square$$

We now consider

$$T = g(U) = \sqrt{U}, \tag{5}$$

with U from (4) as an estimator of γ. According to Theorem 2 the estimator (5) is asymptotically normally distributed with mean γ and variance $(\gamma^2/2 + \gamma^4)/\sum_{i=1}^{k}(n_i - 1)$.

Theorem 2 *Let γ be a population coefficient of variation that is shared by k populations, as defined by Definition 1, and let $v = \sum_{i=1}^{k}(n_i - 1)$. Let $T_v = \sqrt{U_v}$, where $U_v = U$ as defined by (4). Then*

$$\sqrt{v}\ (T_v - \gamma) \xrightarrow{d} N(0\,, \gamma^2/2 + \gamma^4), \quad \text{as } v \to \infty. \tag{6}$$

Proof. By Theorem 1, $\sqrt{v}\ U_v \xrightarrow{d} \mathrm{N}(\gamma^2\,, 2\gamma^4 + 4\gamma^6)$ as $v \to \infty$. Then, by application of Theorem 3.1A in Serfling (1980),

$$\sqrt{v}\ T_v = \sqrt{v}\ f(U_v) \xrightarrow{d} \mathrm{N}(f(\gamma^2)\,, (f'(\gamma^2))^2(2\gamma^4 + 4\gamma^6)), \quad \text{as } v \to \infty,$$

and (6) follows since $(f'(\gamma^2))^2(2\gamma^4 + 4\gamma^6) = \gamma^2/2 + \gamma^4$. $\square$

The expected values of the coefficients of variation c_i are not defined since the densities of m_i, $i = 1, 2, \ldots, k$, are positive in neighborhoods of zero. As a consequence the expected value of the estimator T, as defined by (5), does not exist. Nevertheless, when the expected values μ_i are sufficiently large, say larger than $3\,\sigma_i$, the probability of averages m_i close to zero are negligible in most applications, and we expect c_i, $i = 1, 2, \ldots, k$, to be close to γ. According to Theorem 2 the estimator T is asymptotically unbiased. We shall now derive a bias correction for the case of small sample sizes.

By a second order series expansion of T, as a function of $\{m_i\,, s_i^2\}$, $i = 1, 2, \ldots, k$,

$$\begin{aligned} E(T) &\approx \gamma + \frac{1}{2}\sum_{i=1}^{k}\left(\frac{\partial^2 T}{\partial (s_i^2)^2} Var(s_i^2) + \frac{\partial^2 T}{\partial m_i^2} Var(m_i)\right) \\ &= \gamma + \frac{1}{2}\sum_{i=1}^{k}\left(\frac{-(n_i-1)^2}{4v^2\gamma^3\mu_i^4}\ \frac{2\sigma_i^4}{n_i-1} + \frac{(n_i-1)\,\gamma}{v\mu_i^2}\left(3 - \frac{n_i-1}{v}\right)\frac{\sigma_i^2}{n_i}\right) \\ &= \gamma - \frac{\gamma}{4v} + \frac{\gamma^3}{2v}\sum_{i=1}^{k}\frac{n_i-1}{n_i}\left(3 - \frac{n_i-1}{v}\right), \end{aligned} \tag{7}$$

where the partial second derivatives are evaluated at $\{m_i\,, s_i^2\} = \{\mu_i\,, \sigma_i^2\}$, $i = 1, 2, \ldots, k$, and $v = \sum_{i=1}^{k}(n_i - 1)$. By (7), we expect T to be close to $\gamma(1 - 1/(4v))$ when γ is small. Thus, a bias adjusted estimator of a population coefficient of variation γ that is shared by k populations is given by

$$\hat{\gamma} = \left(1 - \frac{1}{4\sum_{i=1}^{k}(n_i-1)}\right)^{-1}\sqrt{\frac{\sum_{i=1}^{k}(n_i-1)\,c_i^2}{\sum_{i=1}^{k}(n_i-1)}}\ ,$$

with notations as in Definition 1.

3 Confidence interval and test for a coefficient of variation that is shared by k populations

In this section we consider the problems of making a confidence interval for a coefficient of variation γ that is shared by k populations and testing the statistical hypothesis that γ equals some specific value γ_0. We suggest that McKay's approximation, as defined by Definition 2, is utilized for these purposes.

Definition 2 *McKay's approximation for the coefficient of variation of sample i, as defined by Definition 1, is given by*

$$K_i = \left(1 + \frac{1}{\gamma^2}\right) \frac{(n_i - 1)c_i^2}{1 + c_i^2(n_i - 1)/n_i} \ . \tag{8}$$

The distribution of McKay's approximation (8) is known to be well approximated by a central χ^2 distribution with $n_i - 1$ degrees of freedom, provided that $\gamma < 1/3$.

The distribution of

$$u_i = \frac{c_i^2}{1 + c_i^2(n_i - 1)/n_i} \ , \qquad i = 1, 2, \ldots, k, \tag{9}$$

is consequently well approximated by a distribution with expected value

$$\theta = \frac{\gamma^2}{1 + \gamma^2} = \gamma^2 - \gamma^4 + \gamma^6 \ldots \tag{10}$$

and variance inversely proportional to $n_i - 1$.

Since the distribution of McKay's approximation K_i is, approximately, central χ^2 distributed with $n_i - 1$ degrees of freedom, $\sum_{i=1}^k K_i$ is, approximately, central χ^2 distributed with $\sum_{i=1}^k (n_i - 1)$ degrees of freedom. Thus $\sum_{i=1}^k K_i = \sum_{i=1}^k (n_i - 1)u_i/\theta$ could be used as an approximate pivotal quantity for constructing a confidence interval for θ as defined by (10). This approximate $100(1 - \alpha)\%$ confidence interval for θ can be written

$$\left[\frac{\sum_{i=1}^k (n_i - 1)u_i}{\chi^2_{1-\alpha/2}} \ , \ \frac{\sum_{i=1}^k (n_i - 1)u_i}{\chi^2_{\alpha/2}}\right],$$

where χ^2_α denotes the 100α:th percentile of a central χ^2 distribution with $\sum_{i=1}^k (n_i - 1)$ degrees of freedom, and u_i is defined by (9). The corresponding approximate $100(1 - \alpha)\%$ confidence interval for γ is

$$\left[\sqrt{\frac{\sum_{i=1}^k (n_i - 1)u_i}{\chi^2_{1-\alpha/2} - \sum_{i=1}^k (n_i - 1)u_i}} \ , \ \sqrt{\frac{\sum_{i=1}^k (n_i - 1)u_i}{\chi^2_{\alpha/2} - \sum_{i=1}^k (n_i - 1)u_i}}\ \right]. \tag{11}$$

Consider the statistical null hypothesis H$_0$: $\gamma = \gamma_0$. This hypothesis is equivalent to the hypothesis H$_0$: $\theta = \theta_0$, where $\theta_0 = \gamma_0^2/(1 + \gamma_0^2)$. Thus we can use

$$\frac{\sum_{i=1}^k (n_i - 1)u_i}{\theta_0} \tag{12}$$

as an approximately central χ^2 distributed, with $\sum_{i=1}^{k}(n_i - 1)$ degrees of freedom, test statistic of the hypothesis H_0: $\gamma = \gamma_0$.

The proposed confidence interval (11) and test (12) rely on the adequacy of McKay's approximation. Since we are interested in the adequacy, we end this section with an investigation of the first two moments of the approximation, as functions of the population coefficient of variation γ and the sample size n.

Theorem 3 *Let γ be the coefficient of variation as defined by Definition 1, and let $K = K_1$ be McKay's approximation for the coefficient of variation in a sample with $n = n_1$ observations, as defined by Definition 2. The first and second moments of McKay's approximation are*

$$\begin{aligned} E(K) &= \frac{n(n-1)\,h(\gamma,n)}{2\theta}\ , \\ E(K^2) &= \frac{n(n^2-1)(n(1+\gamma^2)\,h(\gamma,n) - 2\gamma^2)}{4\theta^2}\ , \end{aligned}$$

where $\theta = \gamma^2/(1+\gamma^2)$ and, for $t = 0, 1, 2\ldots$,

$$h(\gamma,n) = \begin{cases} \gamma^2(1-\exp(-1/\gamma^2)), & n = 2 \\ \displaystyle\sum_{r=0}^{t}(-1)^r\left(\frac{\gamma^2}{t+3/2}\right)^{r+1}\frac{\Gamma(t+3/2)}{\Gamma(t+3/2-r)} \\ \displaystyle +(-1)^{t+1}\left(\frac{\gamma^2}{t+3/2}\right)^{t+3/2}\frac{2\,\Gamma(t+3/2)}{\sqrt{\pi}}\ d\left(\frac{\sqrt{t+3/2}}{\gamma}\right), & n = 3+2t \\ \displaystyle\sum_{r=0}^{t}(-1)^r\left(\frac{\gamma^2}{t+2}\right)^{r+1}\frac{\Gamma(t+2)}{\Gamma(t+2-r)} \\ \displaystyle +(-1)^{t+1}\left(\frac{\gamma^2}{t+2}\right)^{t+2}\Gamma(t+2)\left(1-\exp(-\frac{t+2}{\gamma^2})\right), & n = 4+2t, \end{cases}$$

with

$$d(x) = \exp(-x^2)\int_0^x \exp(z^2)\ dz.$$

Proof. Forkman and Verrill (2007) showed that $K\theta/n$ is type II noncentral beta distributed with parameters $(n-1)/2$, $1/2$ and n/γ^2. Consequently $X = 1 - K\theta/n$ is type I noncentral beta distributed with parameters $1/2$, $(n-1)/2$ and n/γ^2. Then, according to Marchand (1997),

$$E(X) = 1 - \frac{(n-1)h(\gamma,n)}{2}\ , \tag{13}$$

$$E(X^2) = 1 - \frac{(n^2-1)\,\gamma^2}{2n} + \frac{(n-1)(n-3+(n+1)\gamma^2)\,h(\gamma,n)}{4}, \tag{14}$$

and the theorem follows since $E(K) = n(1 - E(X))/\theta$ and $E(K^2) = n^2(1 - 2E(X) + E(X^2))/\theta^2$. $\square$

The function d, required for odd sample sizes in Theorem 3, is Dawsons's integral, which has been tabulated by Abramowitz and Stegun (1972). Since $\exp(-(t+2)/\gamma^2) \approx 0$, $t = 0, 1, 2 \ldots$, Theorem 3 makes it easy to calculate approximate first and second moments, especially for even sample sizes. For example,

$$E(K) \approx \begin{cases} 1\,(1+\gamma^2), & n=2 \\ 3\left(1+\dfrac{\gamma^2}{2}-\dfrac{\gamma^4}{2}\right), & n=4 \\ 5\left(1+\dfrac{\gamma^2}{3}+\dfrac{2\gamma^4}{9}+\dfrac{8\gamma^6}{9}\right), & n=6, \end{cases}$$

and

$$E(K^2) \approx \begin{cases} 3\,(1+2\gamma^2+\gamma^4), & n=2 \\ 15\,(1+\gamma^2-\gamma^4-\gamma^6), & n=4 \\ 35\left(1+\dfrac{8\gamma^2}{3}+5\gamma^4+6\gamma^6+\dfrac{8\gamma^8}{3}\right), & n=6. \end{cases}$$

Notice that when γ is small, $n = 2, 4, 6$, $E(K)$ approximately equals $n-1$ and $E(K^2)$ approximately equals $(n-1)(n+1)$, which is the exact first and second moments, respectively, of a χ^2 distributed random variable with $n-1$ degrees of freedom.

4 Test for equality of a coefficient of variation that is shared by k_1 populations and a coefficient of variation that is shared by k_2 populations

We now introduce a statistical test for the hypothesis that a coefficient of variation γ_1 that is shared by k_1 populations equals a coefficient of variation γ_2 that is shared by k_2 populations. Definition 3 makes clear the setting and what we mean by coefficients of variation in this case.

Definition 3 *Let $y_{rij} = \mu_{ri} + e_{rij}$, where e_{rij} are independently distributed $N(0, \sigma_{ri}^2)$, $r = 1, 2$, $i = 1, 2, \ldots, k_r$ and $j = 1, 2, \ldots, n_{ri}$, with positive expected values μ_{ri} and positive population coefficients of variation $\gamma_r = \sigma_{ri}/\mu_{ri}$. The* *coefficient of variation c_{ri}, $r = 1, 2$, $i = 1, 2, \ldots, k_r$, is defined as $c_{ri} = s_{ri}/m_{ri}$, where m_{ri} and s_{ri} are*

$$m_{ri} = \frac{1}{n_{ri}} \sum_{j=1}^{n_{ri}} y_{rij} \qquad \textit{and} \qquad s_{ri} = \sqrt{\frac{1}{n_{ri}-1} \sum_{j=1}^{n_{ri}} (y_{rij} - m_{ri})^2}\ , \tag{15}$$

respectively.

4.1 A test for equality of coefficients of variation

Consider the hypothesis H$_0$: $\gamma_1 = \gamma_2$. Let

$$u_{ri} = \frac{c_{ri}^2}{1 + c_{ri}^2 (n_{ri} - 1)/n_{ri}}\ , \qquad r = 1, 2,\ i = 1, 2, \ldots, k_r,$$

and

$$\theta_r = \frac{\gamma_r^2}{1 + \gamma_r^2}\ , \qquad r = 1, 2,$$

with notation according to Definition 3. Since $\sum_{i=1}^{k_r} (n_{ri} - 1) u_{ri}/\theta_r$ is approximately central χ^2 distributed with $\sum_{i=1}^{k_r} (n_{ri} - 1)$ degrees of freedom, and since $\theta_1 = \theta_2$ when the hypothesis is true,

$$F = \frac{\sum_{i=1}^{k_1} (n_{1i} - 1) u_{1i} / \sum_{i=1}^{k_1} (n_{1i} - 1)}{\sum_{i=1}^{k_2} (n_{2i} - 1) u_{2i} / \sum_{i=1}^{k_2} (n_{2i} - 1)} \tag{16}$$

is approximately F distributed with $\sum_{i=1}^{k_1} (n_{1i} - 1)$ and $\sum_{i=1}^{k_2} (n_{2i} - 1)$ degrees of freedom. Thus F could be used as an approximately F distributed test statistic for the hypothesis of equal coefficients of variation.

When $k_1 = k_2 = 1$, the test statistic F, as defined by (16), simplifies to

$$F = \frac{c_1^2/(1 + c_1^2 (n_1 - 1)/n_1)}{c_2^2/(1 + c_2^2 (n_2 - 1)/n_2)}\ , \tag{17}$$

where $c_r = c_{r1}$ and $n_r = n_{r1}$, r = 1, 2, as defined by Definition 3. According to Theorem 4 the distribution of the logarithm of F, as defined by (17), equals the distribution the logarithm of an F distributed random variable plus some error variables that are in probability of small orders.

Theorem 4 *Let $\gamma = \gamma_1 = \gamma_2$ as defined by Definition 3, with $k_1 = k_2 = 1$. Let X be an F distributed random variable with $n_1 - 1$ and $n_2 - 1$ degrees of freedom, let Z be a standardized normal random variable, and let U_1 and U_2*

be χ^2 distributed random variables with $n_1 - 1$ and $n_2 - 1$ degrees of freedom, respectively. Let X, Z, U_1 and U_2 be independent. Then

$$\log F \overset{d}{=} \log X + 2Z\gamma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} + \left(\frac{U_1}{n_1} - \frac{U_2}{n_2}\right)\gamma^2 + R(n_1, n_2, \gamma), \tag{18}$$

where F is defined by (17) and

$$R(n_1, n_2, \gamma) = O_p(\max\{n_1^{-1}\gamma^2, n_2^{-1}\gamma^2, \gamma^4\}).$$

Proof. Write $\log F$ as

$$\log F = \log c_1^2\left(1 + \frac{(n_1-1)c_1^2}{n_1}\right)^{-1} - \log c_2^2\left(1 + \frac{(n_2-1)c_2^2}{n_2}\right)^{-1}. \tag{19}$$

Let $W_r = U_r/(n_r - 1)$, $r = 1, 2$, and let Z_1 and Z_2 be independent standard normal random variables. The distributions of the averages m_{r1} and the standard deviations s_{r1}, $r = 1, 2$, as defined by Definition 3, equals the distributions of $\mu_{r1} + Z_r\sigma_{r1}/\sqrt{n_r}$ and $\mu_{r1}\gamma\sqrt{W_r}$, respectively. Thus c_r^2 equals $W_r\gamma^2/(1 + Z_r\gamma/\sqrt{n})^2$ in distribution. The distribution of the first term in (19) consequently equals the distribution of

$$\begin{aligned}
&\log W_1\gamma^2 - \log\left(1 + \frac{2Z_1\gamma}{\sqrt{n_1}} + \frac{Z_1^2\gamma^2}{n_1} + \frac{(n_1-1)W_1\gamma^2}{n_1}\right) \\
&= \log W_1\gamma^2 + \frac{2Z_1\gamma}{\sqrt{n_1}} + \frac{Z_1^2\gamma^2}{n_1} + \frac{(n_1-1)W_1\gamma^2}{n_1} \\
&\quad -\frac{1}{2}\left(\frac{2Z_1\gamma}{\sqrt{n_1}} + \frac{Z_1^2\gamma^2}{n_1} + \frac{(n_1-1)W_1\gamma^2}{n_1}\right)^2 + \ldots \\
&= \log W_1\gamma^2 + \frac{2Z_1\gamma}{\sqrt{n_1}} + \frac{(n_1-1)W_1\gamma^2}{n_1} + O_p\left(\max\{\frac{\gamma^2}{n_1}, \gamma^4\}\right),
\end{aligned} \tag{20}$$

where O_p denotes order in probability, defined as in Azzalini (1996). The corresponding calculations can be made also for the second term in (19), and (18) follows. □

4.2 Simulation study

In this section we investigate, by Monte Carlo technique, the significance levels and powers of the introduced approximate F-test (17), for the hypothesis H_0: $\gamma_1 = \gamma_2$ when $k_1 = k_2 = 1$. We also study the likelihood ratio test, Bennett's test as modified by Shafer and Sullivan (1986), and Miller's test. These tests are, for quick reference, given in the Appendix.

In each simulation two samples with n_1 and n_2 observations, respectively, were randomly generated 20 000 times in Release 13 of MATLAB (The Mathworks Inc., Natick, MA, USA). The observations belonged to normal distributions with expected values 100 and 1000, and with coefficients of variation γ_1 and γ_2, respectively. The tests were performed with significance level 5% against the alternative hypothesis of unequal coefficients of variation, i.e. the tests were two-sided.

Four cases were studied. The type I errors of the tests were investigated in Case 1–3, and the powers of the tests were investigated in Case 4. The first case had small coefficients of variation (5%) and equal sample sizes. The second case had large coefficients of variation (25%) and equal sample sizes. The third case had large coefficients of variation (25%) and unequal sample sizes. In the fourth case one coefficient of variation was 10% and the other 5%, and the sample sizes were equal. In the balanced cases, where $n = n_1 = n_2$, (Case 1, 2 and 4), the sample sizes 2–20 were investigated, i.e. 19 simulations were made per case. In the unbalanced cases (Case 3), $n_1 = 4$ but n_2 varied from 2 to 20. Thus 19 simulations were made also for Case 3.

The results of the simulation study are presented in Figure 1–4, with one figure for each investigated case. The likelihood ratio test showed too large frequency of type I error (Figure 1–3), especially for sample sizes smaller than 10. When $n_1 = 2$ the frequency of rejected hypotheses was larger than 20% with the likelihood ratio test (Case 1–3). Bennett's test was also too liberal, though not as liberal as the likelihood ratio test (Figure 1–3). Miller's test performed better than the likelihood ratio test and Bennett's test with regard to type I error, but had problems when $n_1 = 2$ (Figure 1–3). The approximate F-test, introduced in this paper, was the only test that produced almost correct frequency of rejected hypotheses in all cases (Figure 1–3). The likelihood ratio test and Bennett's test, which were too liberal, showed better power for small sample sizes than Miller's test and the approximate F-test (Figure 4).

5 Discussion

In applications with constant, or almost constant, coefficients of variation it is usually appropriate to assume that the data follows lognormal distributions. One should thus always consider working on the log scale (Cole, 2000). After log transformation of the data the usual tests for equality of variances, such as Bartlett's test, could be applied. It is, however, not always appropriate to assume that the data is lognormally distributed. In immunoassays, for example, normally distributed errors in the volumes of samples could result in normally

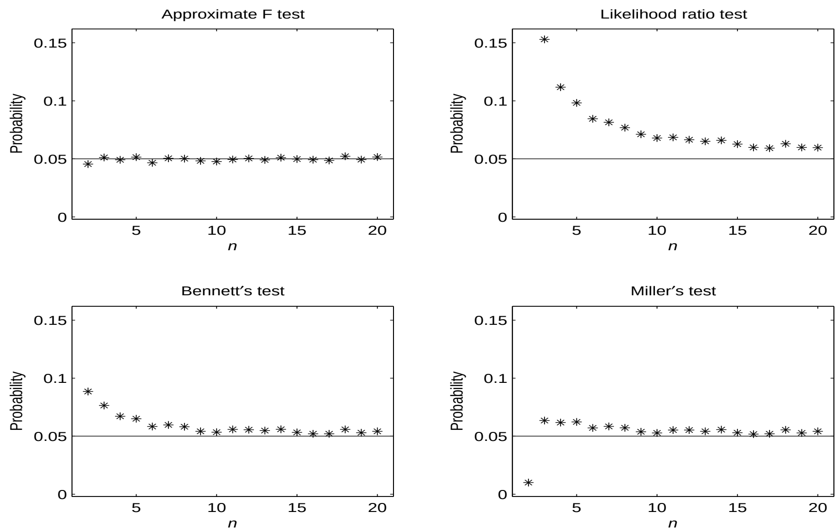

Figure 1: Case 1. Probability of type I error when $\gamma_1 = \gamma_2 = 5\%$ and $n = n_1 = n_2$. Significance level 5%.

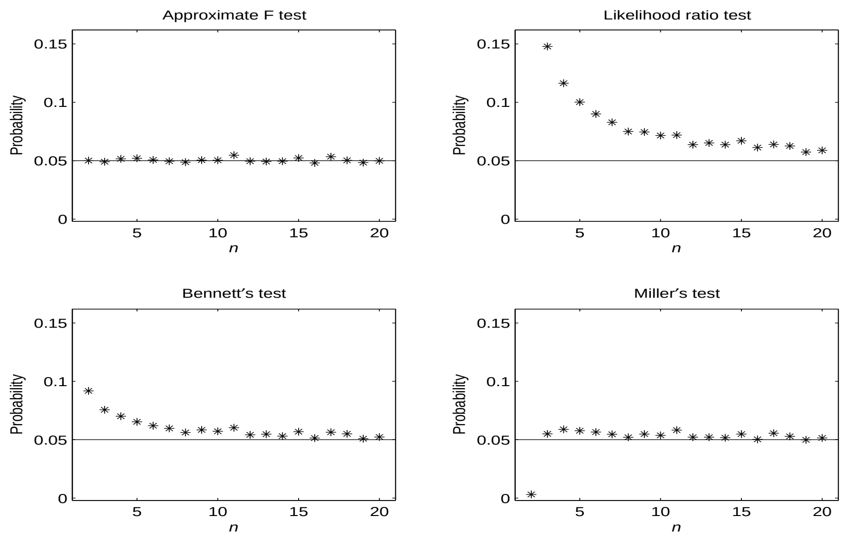

Figure 2: Case 2. Probability of type I error when $\gamma_1 = \gamma_2 = 25\%$ and $n = n_1 = n_2$. Significance level 5%.

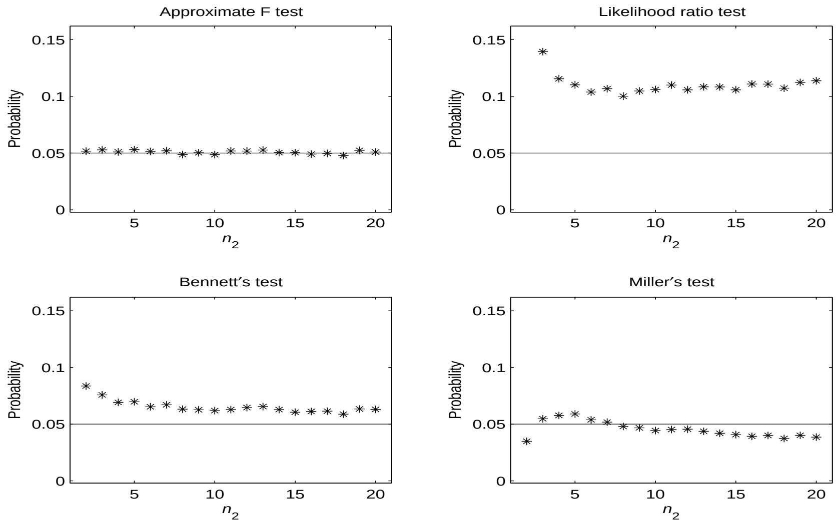

Figure 3: Case 3. Probability of type I error when $\gamma_1 = \gamma_2 = 25\%$ and $n_1 = 4$. Significance level 5%.

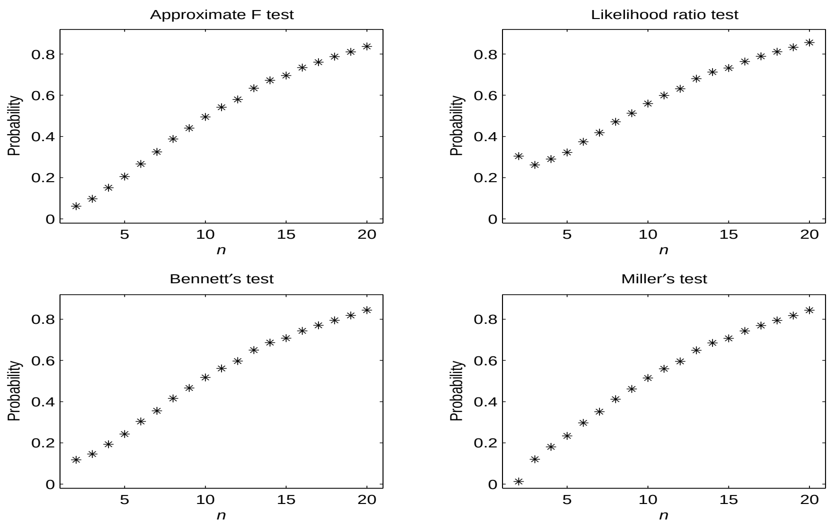

Figure 4: Case 4. Power when $\gamma_1 = 5\%, \gamma_2 = 10\%$ and $n = n_1 = n_2$. Significance level 5%.

distributed measurements of concentration with constant coefficient of variation. In this paper we have discussed inference for the coefficient of variation when there are reasons to believe that the data is normally distributed.

Since it is often necessary to relate the standard deviation to the level of the measurements, the coefficient of variation is a widely used measure of dispersion. Coefficients of variation are often calculated on samples from several independent populations, and questions about how to compare them naturally arise. There is a need for an explicit estimator of a population coefficient of variation that is shared by several populations. Such an estimator has been given in this paper.

For making confidence intervals we have considered McKay's approximation, which is valid only when the population coefficient of variation γ is smaller than 1/3. Coefficients of variation are usually calculated on positive data, such as measurements of concentration, weight or length. Given that the positive measurements are approximately normally distributed γ is smaller than 1/3, because otherwise the expected value is smaller than 3 standard deviations, and the probability of negative observations is not negligible.

Over the years many tests have been proposed for equality of coefficients of variation. In this paper an additional test has been introduced: the approximate F-test. Unlike many other tests the new test could be applied not only when there are one estimate per population coefficient of variation, but also when there are several. The small simulation study reported in this paper indicated good performance of the approximate F-test, especially with regard to type I error. It would, however, be interesting to see results from a larger simulation study, including more cases, several tests and an investigation of robustness. As already pointed out, the methods proposed in this paper are intended for normally distributed data. Miller (1991b) suggested a nonparametric test for equality of coefficients of variation that is useful for other distributions and recommended by Fung and Tsang (1998).

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Appendix: The likelihood ratio, Bennett's, and Miller's test

Let $m_r = m_{r1}$ and $c_r = c_{r1}$ denote the average and the coefficient of variation, respectively, in the r:th sample, $r = 1, 2$, as defined by Definition 3.

The likelihood ratio test statistic can be written as

$$-2\log\lambda = n_1 \log\frac{n_1(\tilde{\gamma}\tilde{\mu_1})^2}{(n_1-1)\ c_1^2 m_1^2} + n_2 \log\frac{n_2(\tilde{\gamma}\tilde{\mu_2})^2}{(n_2-1)\ c_2^2 m_2^2}\ , \tag{21}$$

where λ is the likelihood ratio and $\tilde{\gamma}$, $\tilde{\mu_1}$ and $\tilde{\mu_2}$ are the maximum likelihood estimates of γ, μ_1 and μ_2, respectively. These are, according to Gerig and Sen (1980),

$$\tilde{\mu}_1 = \frac{n_1 m_1 \tilde{\mu}_2}{(n_1+n_2)\tilde{\mu}_2 - n_2 m_2}, \quad \tilde{\mu}_2 = -\frac{q}{2p} + \sqrt{\frac{q^2}{4p^2} - \frac{r}{p}}$$

and

$$\tilde{\gamma} = \frac{1}{\tilde{\mu}_2}\sqrt{\frac{(n_2-1)\ c_2^2 m_2^2}{n_2} + m_2^2 - m_2\tilde{\mu}_2}, \tag{22}$$

where

$$\begin{aligned} p &= (n_1+n_2)c_1^2 + n_2, \\ q &= -(2n_2c_1^2 + 2n_2 - n_1)m_2, \\ r &= \frac{(n_2^2(c_1^2+1) - n_1^2(c_2^2+1))m_2^2}{n_1+n_2}. \end{aligned}$$

Asymptotically (21) is χ^2 distributed with 1 degree of freedom.

Bennett's test statistic, as modified by Shafer and Sullivan (1986), can be written as

$$\begin{aligned} (n_1+n_2-2)\log\left(\frac{1}{n_1+n_2-2}\left(\frac{(n_1-1)c_1^2}{1+c_1^2(n_1-1)/n_1} + \frac{(n_1-1)c_1^2}{1+c_1^2(n_1-1)/n_1}\right)\right) \\ -(n_1-1)\log\left(\frac{(n_1-1)c_1^2}{(n_1-1)(1+c_1^2(n_1-1)/n_1}\right) \\ -(n_2-1)\log\left(\frac{(n_2-1)c_2^2}{(n_2-1)(1+c_2^2(n_2-1)/n_2)}\right), \end{aligned}$$

which should be compared with a χ^2 distribution with 1 degree of freedom.

Miller's test statistic (Miller, 1991), which should be compared with a standard normal distribution, is

$$(c_1-c_2)\left(\frac{c^2}{2(n_1-1)} + \frac{c^4}{n_1-1} + \frac{c^2}{2(n_2-1)} + \frac{c^4}{n_2-1}\right)^{-1/2},$$

where $c = ((n_1-1)c_1 + (n_2-1)c_2)/(n_1+n_2-2)$.