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Abstract

In this paper, we discuss the efficiency of noise reduction for curve fitting in growth curve models. We use singular spectrum analysis as a nonlinear–nonparametric denoising method. A set of longitudinal measurements is used in considering the performance of the method. The results show that noise reduction is important for curve fitting in growth curve model and also, that the singular spectrum analysis technique can be used as a powerful tool for noise reduction in longitudinal measurements.

Keywords: Growth curve models, curve fitting, noise reduction, singular spectrum analysis.

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1 Introduction

Growth is a fundamental property of biological systems, occurring at the populations level, as well as within organisms. Much research has been devoted to modeling growth processes, and there are many ways of doing this, including: mechanistic models, time series, stochastic differential equations etc. Sometimes we simply wish to summarize growth observations in terms of a few parameters, perhaps in order to compare individuals or groups.

The growth curve model is used for the analysis of longitudinal data, in which measurements are repeatedly taken on a response variable at a number of time points. This method enables us to investigate an overall pattern of change in the response variable over time. It also allows us to examine the effects of time-invariant explanatory variables on the temporal pattern of the response variable. Let t be the age at examination and y_t be the measurement at the age t. We assume a statistical model such as $y_t = H(t) + \epsilon_t$ $(t = 1, \ldots, T)$, where the random variable ϵ_t has mean 0 and variance σ_t^2. $H(t)$ is called the growth curve.

Analysis of growth data sets is associated with numerous sources of error that affect its results and reduce its usefulness. It is well known that errors can seriously limit the performance of the methods and techniques. Effective methods for dealing with noisy data, especially noisy time series are currently still lacking.

There are two main approaches for fitting a model to a noisy time series. According to the first one, ignoring the presence of noise, we fit a model directly from noisy data. According to the second approach, which is often more effective that the first one, we start by filtering the noisy time series in order to reduce the noise level and then fit a model to noise-reduced data.

In the case of the former approach, we fit the function H on the noisy data. In this sense the curve fitting must be bad if the noise level is relatively high, especially for the growth curve model whose variance is a function of time t. Assume the noise-reduced time series has been obtained by some noise reduction method, e.g., a singular spectrum analysis technique (SSA) and denote it by $Z_T = (z_1, \ldots, z_T)$ and $y_t = z_t + \omega_t$ $(t = 1, \ldots, T)$, where the term ω_t is the noise which was removed by the noise reduction method. The ideal result of noise reduction is $z_t = H(t)$ or $\omega_t = \epsilon_t$ for each t. Then we fit a growth curve model on the noise-reduced time series. In fact, the fitted model on the noise-reduced time series should be optimal, because we remove the noise term ω_t from original series. If the noise has been significantly reduced in the noisy time series, then the latter approach is expected to give much better results than the former approach.

The problem of filtering time series to obtain noiseless series with minimum loss of information has been widely studied. There are several nonlinear noise reduction methods such as a local projective method, digital butterworth filters, splines, filters based on spectral analysis, singular value decomposition (SVD) and simple nonlinear filtering. It is currently accepted that SVD-based methods are more effective than the others for noise reduction in deterministic time series (Soofi and Cao, 2002). SSA naturally incorporates the filtering of the series and the SVD. Recent research shows that SSA can be used as an alternative to traditional digital filtering methods. For example, Alonsoa et al. (2004) showed the superiority of the SSA technique over traditional methods used in biomechanical analysis for filtering data.

The paper, besides this Introduction, contains five sections and is organized as follows. In section 2, we provide necessary theoretical background and concisely describe the growth curve model used here. In Section 3, we present the SSA technique as a non-parametric nonlinear noise reduction method. In Section 4, we discuss data and parameter estimation. The results are presented in Section 5. Finally some conclusions are given in Section 6.

2 Growth curve model

The growth curve model (GCM) was introduced by Potthoff and Roy (1964) and subsequently studied among others by Rao (1965). The first research considering growth curves was presented by Wishart (1938) and discrimination between growth curves was discussed by Burnaby (1966). Wishart (1938) recommended that a general regression model should be fitted to each curve and that the effects of the experimental treatments should be evaluated by analyzing the coefficients in the model. Since then different aspects of the model has been considered by many authors including Khatri (1966), Krishnaiah (1969), Gleser and Olkin (1970), Srivastava and Khatri (1979), von Rosen (1989). See also von Rosen (1991) and Kollo and von Rosen (2005) for the review of the models. The growth curve model is defined as:

$$\mathbf{Y}_{T\times n} = \mathbf{D}_{T\times m}\mathbf{B}_{m\times r}\mathbf{Z}_{r\times n} + \epsilon_{T\times n} \tag{1}$$

where $\mathbf{D}$ and $\mathbf{Z}$ are known design matrices of rank $m < r$ and $r < n$, respectively, and the regression coefficients $\mathbf{B}$ are unknown. Furthermore, the columns of the error matrix ϵ are independent p-variate normal with mean zero and common unknown covariance matrix $\sum$, that is $\mathbf{Y} \sim N_{T\times n}(\mathbf{DBZ}, \sum, \mathbf{I}_n)$. Usually, T is the number of time series points observed on each of n cases, $m-1$ is the degree of polynomial in time, and r is the number of treatment

groups. Here we use a univariate growth curve model (for more information about multivariate and extended GCM see von Rosen (1989)). It should be noted that, although historically growth curve models refer to fitting individual curves, these have been widely accepted as those using in mixed models (for more information, see for example, Demidenko (2004)).

The idea of using principal components to summarize the major sources of variation in a set of growth curves dates back to Rao (1958), and several examples in Ramsay and Silverman (1997) are of this type. Analysis of GCM is often concerned with predicting future growth, and one way of doing this is to use principal components as predictors (Jolliffe, 2002). A form of generalized principal component regression was developed for this purpose by Rao (1987). Growth curve can be considered as a special case of longitudinal data. Berkey et al. (1991) used principal components to model longitudinal data, the model they used is called a longitudinal principal component model.

We consider three models for analysis of our data sets as used by Hassani et al. (2003). Jenss and Bayley (1937) proposed the following growth curve:

$$y_t = A + Bt - \exp(C - Dt). \tag{2}$$

Here we use a special case of (2) in the following form:

$$y_t = \alpha_1 + \beta_1 t - \gamma_1 \exp(1 - \delta_1 t). \tag{3}$$

The von Bertalanffy model (von Bertalanffy, 1957) sometimes called the Borody-Bertallanffy model, was the first growth model especially designed to describe individual growth:

$$y_t = A - (1 - \exp(B - Ct)). \tag{4}$$

Here we use the following version of (4):

$$y_t = \alpha_2 - \beta_2 \exp(1 - \gamma_2 t). \tag{5}$$

Count (1943) proposed a growth pattern of the human physique as $A + Bt + C\,ln\,(t)$. Here we use the following modification of Count's model (Shohoji and Sasaki, 1987):

$$y_t = \alpha_3 + \beta_3 t + \gamma_3 \log(1 + \delta_3 t). \tag{6}$$

3 Singular spectrum analysis

The Singular Spectrum Analysis (SSA) technique is a powerful technique of time series analysis incorporating the elements of classical time series analysis,

multivariate statistics, multivariate geometry, dynamical systems and signal processing. The main purpose of SSA is to decompose the original series into a sum of a small number of time series, so that each subseries can be identified as either a trend, periodic or quasi-periodic component (perhaps, amplitude-modulated), or noise. This is followed by a reconstruction of the original series.

The SSA technique consists of two complementary stages: decomposition and reconstruction. At the first stage we decompose the time series and at the second stage we reconstruct the original time series and use the reconstructed time series for forecasting. Here we provide a brief discussion on the methodology of the basic SSA method; see Golyandina et al. (2001) and Hassani (2007) for more information and many variations of the basic SSA.

Short description of the basic SSA

The main idea of the basic SSA is as follows. Consider the real-valued nonzero time series $Y_T = (y_1, \ldots, y_T)$ of sufficient length T. Let $K = T - L + 1$, where L is some integer called the window length (we can assume $L \leq T/2$). Define the so-called 'trajectory matrix' $\mathbf{X}$:

$$\mathbf{X} = (x_{ij})_{i,j=1}^{L,K} = \begin{pmatrix} y_1 & y_2 & y_3 & \cdots & y_K \\ y_2 & y_3 & y_4 & \cdots & y_{K+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_L & y_{L+1} & y_{L+2} & \cdots & y_T \end{pmatrix} \tag{7}$$

Obviously $x_{ij} = y_{i+j-1}$ so that the matrix $\mathbf{X}$ has equal elements on the diagonals $i + j$ =const. We then consider $\mathbf{X}$ as a multivariate data with L characteristics and $K = T - L + 1$ observations. The columns X_j of $\mathbf{X}$, considered as vectors, lie in an L-dimensional space $\mathbb{R}^L$. Define the matrix $\mathbf{X}\mathbf{X}^T$. Singular value decomposition (SVD) of $\mathbf{X}\mathbf{X}^T$ provides us with the collections of L eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_L \geq 0$ and the corresponding eigenvectors $U_1, U_2, \ldots, U_L$, where U_i is the normalised eigenvector corresponding to the eigenvalue λ_i $(i = 1, \ldots, L)$.

The SVD of the trajectory matrix can be written as:

$$\mathbf{X} = \mathbf{E}_1 + \cdots + \mathbf{E}_d, \tag{8}$$

where $\mathbf{E}_i = \sqrt{\lambda_i} U_i V_i^T$ $(i = 1, \ldots, d)$, d is the number of non-zero eigenvalues of $\mathbf{X}\mathbf{X}^T$, and $V_1, \ldots, V_d$ are the principal components defined as $V_i = \mathbf{X}^T U_i / \sqrt{\lambda_i}$.

The collection $(\sqrt{\lambda_i}, U_i, V_i)$ is referred to as the i-the eigentriple of the matrix $\mathbf{X}$.

A group of r (with $1 \leq r \leq d$) eigenvectors determines an r-dimensional hyperplane in the L-dimensional space $\mathbb{R}^L$ of vectors X_j. The L_2-distance between vectors $X_j \in \mathbb{R}^L$ and this r-dimensional hyperplane is equal to $\sum_{j \notin I} \lambda_j$ and can be rather small which would mean that $\tilde{\mathbf{X}}$, the projection of $\mathbf{X}$ into this hyperplane, approximates well the original matrix $\mathbf{X}$. Subsequent averaging over the diagonals of $\tilde{\mathbf{X}}$ allows us to obtain a series that can be considered as an approximation to the original series.

Selection of parameters

Here we consider a version of SSA where we split the set if indicies $\{1, 2, \ldots, d\}$ into two groups only: $I = \{1, \ldots, r\}$ and $\bar{I} = \{r+1, \ldots, d\}$. We associate the group I (and the related matrix $\mathbf{E}_I = \mathbf{E}_1 + \ldots + \mathbf{E}_r$) with signal and the group $\bar{I}$ with noise. The SSA method requires then the selection of two parameters, the window length L and the number of elementary matrices r. There are specific rules for selecting these parameters; their choice depends on structure of the data and the analysis we want to perform. Detailed description of parameter selection procedures is given in Golyandina et al. (2001). Here we summarize a few general rules.

The window length L is the single parameter that should be selected at the decomposition stage. Selection of the proper window length depends on the problem in hand, and on preliminary information about the time series. For the series with a complex structure, too large window length L can produce an undesirable decomposition of the series components of interest, which may lead, in particular, to their mixing with other series component. Let us consider the problem of trend extraction in GCM. Since trend is a relatively smooth curve, its separability from noise requires small values of L.

The first elementary matrix $\mathbf{E}_1$ with the norm $\sqrt{\lambda_1}$ has the highest contribution to the norm of $\mathbf{X}$ in (8) and the last elementary matrix $\mathbf{E}_d$ with the norm $\sqrt{\lambda_d}$ has the lowest contribution to the norm of $\mathbf{X}$. The plot of the eigenvalues $\lambda_1, \cdots, \lambda_d$ gives an overall view concerning the values of the eigenvalues and is essential in deciding where to truncate the summation of (8) in order to build a good approximation of the original matrix. A slowly decreasing sequence of eigenvalues typically indicate the presence of noise in the series.

A group of r (with $1 \leq r < L$) eigenvectors determine an r-dimensional hyperplane in the L-dimensional space $\mathbb{R}^L$ of vectors X_j. The distance between vectors X_j ($j = 1, \ldots, K$) and this r-dimensional hyperplane can be

rather small (it is controlled by the choice of the eigenvalues) meaning that the projection of $\mathbf{X}$ into this hyperplane is a good approximation of the original matrix $\mathbf{X}$. If we choose the first r eigenvectors $P_1, \ldots, P_r$, then the squared L_2-distance between this projection and $\mathbf{X}$ is equal to $\sum_{j=r+1}^{L} \lambda_j$. According to the Basic SSA algorithm, the L-dimensional data is projected onto this r-dimensional subspace and the subsequent averaging over the diagonals allows us to obtain an approximation to the original series.

4 Parameter estimation and data

The number of measurements and the age at examination, in the longitudinal human growth study, are usually very different from subject to subject. It is difficult to identify individual growth and to compare growth patterns directly by the original measurement. We can reduce the dimensionality by estimating the growth parameters with minimum loss of information on measurement. We use 120 males' and 108 females' longitudinal measurement records of height from birth to age 3, which accumulated in different clinics in Tehran. They were born from 1994 to 1998 and all have a record of height at birth. Their height was measured occasionally at a clinic center but the age intervals between successive measurements are different in each subject.

The Marquart method was applied for estimating the growth parameter by the least squares method. We used convergent criteria so that no estimates violated any boundary condition of the growth parameters and that the relative amount of correction of successive estimates of each growth parameter was less than 0.00001, simultaneously.

As the number of unknown parameters may be different for each model, it is difficult to compare directly the goodness of fit, among (linear and non-linear) growth models. The Akaike Information Criterion (AIC) (Akaike, 1973), or Schwarz Information Criterion (SIC) for comparing the goodness of fit point of view are:

$$AIC = -2M + 2k, \quad SIC = -2M + 2k \ln(N) \tag{9}$$

where M is the maximum log-likelihood, k is the number of free parameters in the model and N is the number of observations. AIC and SIC are based on two items, the average log-likelihood function and the penalty function. These criteria consider the penalty for the number of free parameter in a model. The growth model minimizing AIC or SIC is chosen as the best model among the others.

5 Results

As we are interested in the extraction of the general tendency of the series, a small value of the window length is sufficient. Here we select $L = 3$. So, based on the window length $L = 3$ and on the SVD of the trajectory matrix, we have 3 eigentriples, ordered by their contributions (shares) into the decomposition. The leading eigentriple describes the general tendency of the series. Since in most cases the eigentriples with small shares are related to the noise component of the series, we need to identify the set of leading eigentriples. Let us consider the result of the SVD step. Figure 1 represents principal components related to the first 3 eigentriples, which are ordered by their contribution (shares) in the decompositions. The next challenge is the selection of the grouping parameter r.

Trend is the slowly varying component of a time series which does not contain oscillatory components. Assume that the time series itself is such a component alone. Practice shows that in this case, one or more of the leading eigenvectors will be slowly varying as well. We know that eigenvectors have (in general) the same form as the corresponding components of the initial time series. Thus we should find slowly varying eigenvectors. It can be done by considering one-dimensional plots of the eigenvectors. From the practical point of view, a natural way of noise extraction is the grouping of the eigentriples, which do not seemingly contain elements of trend or oscillation. Irregular behavior of eigenvectors can indicate that they are part of noise.

Based on the above information and also Figure 1, we have decided that the eigentriples 2 and 3 correspond to the noise. Therefore, we choose the first eigentriple to reconstruct the noise free series and eigentriples 2 and 3 as the noise or residual series. It should be noted that, residuals and influential observations in the growth curve models have been studied extensively producing a huge amount of literature, See, for example, Liski (1991), Pan and Fang (1996), and von Rosen (1995). von Rosen (1995) has shown that residuals in the growth curve models are symmetrically distributed around zero and, has obtained a couple of moment relations for three types of residuals. Hamid and von Rosen (2006) have considered residuals in extended GCM.

We apply the previously mentioned models in Section 2 and the SSA technique with indicated parameters for noise reduction for fitting to the height of 228 Iranian children. Tables 1 and 2 present the fundamental statistics of the growth parameters along with the mean AIC and mean SIC. The first column of Tables 1 and 2 present the model used here, the second column shows the estimated parameters for each model, and columns 3–7 represent fundamental statistics of growth parameters for each model. We also provide the mean AIC

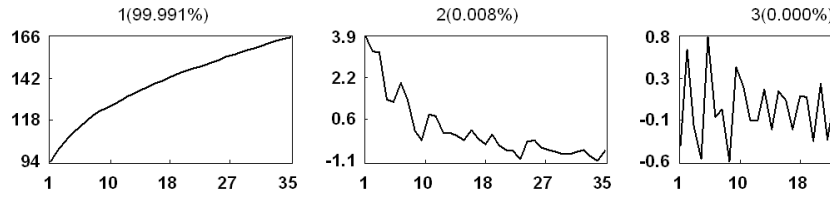

Figure 1: Principal components related to the first 3 eigentriples.

and SIC for each model. The values in brackets are related to noise-reduced time series. As it appears from Tables 1 and 2, the values of AIC, SIC and SD (standard deviation) for noise free series are smaller than the values for noisy series for all models, confirming that noise reduction does help to obtain a better model for fitting used series.

To acquire a better understanding of the efficiency of noise reduction on parameter estimation, we examine the distribution of the estimated parameters for both noisy and noise–free series. Note that under certain regularity conditions, the estimated growth parameters are asymptotically normally distributed and the estimators of the growth parameters are consistent estimators for individual longitudinal data (Shohoji et al., 1991).

Figure 2 shows the Normal distribution of estimated growth parameters α_1, β_1, γ_1 and δ_1 which are obtained by fitting (3) on longitudinal measurements for 120 boys. The same results are obtained for the girls but are not provided here[2]. Confirming the existing results in growth curve model literature, the distribution of the estimated parameters using both noise free and noisy longitudinal measurements are Normal. However, the noise–free longitudinal measurements have a distribution with standard deviation smaller than the noisy one. Figure 3 shows the Normal distribution of the estimated parameters α_2, β_2 and γ_2 and Figure 4 shows Normal distribution of estimated parameters α_3, β_3, γ_3 and δ_3 which are obtained by fitting (5) and (6) on longitudinal measurements for boys, respectively. These Figures confirm that, without any exception, curve fitting on the noise–free series gives a set of growth parameters with less deviation.

6 Conclusion

A set of longitudinal measurements used to answer the question of whether noise reduction matters for curve fitting in growth curve models. The singular

[2]Details of the results for girls are available from authors upon request.

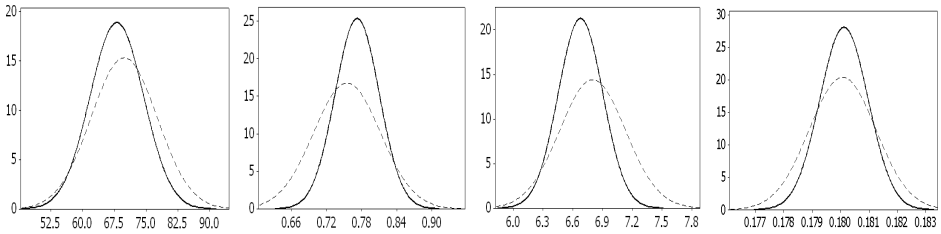

Figure 2: Distribution of estimated parameters of α_1, β_1, γ_1 and δ_1 (left to right) for noisy time series (dashed line) and noise-reduced time series (thick line).

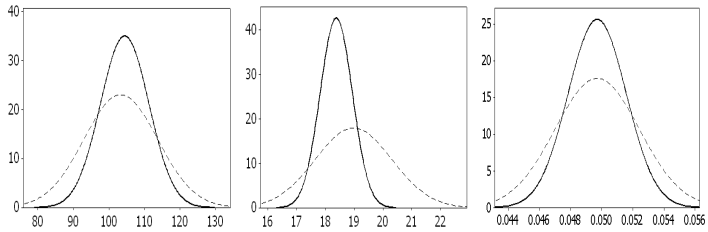

Figure 3: Distribution of estimated parameters of α_2, β_2 and γ_2 (left to right) for noisy time series (dashed line) and noise-reduced time series (thick line).

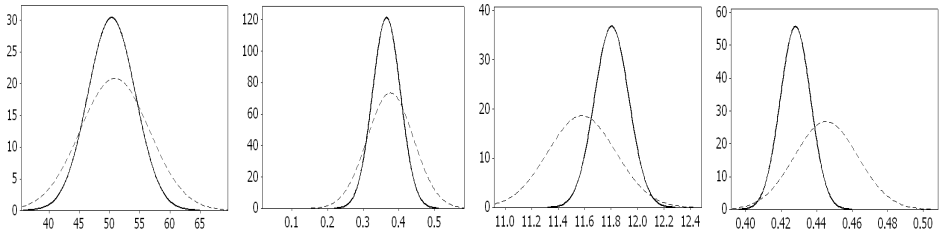

Figure 4: Distribution of estimated parameters of α_3, β_3, γ_3 and δ_3 (left to right) for noisy time series (dashed line) and noise-reduced time series (thick line).

spectrum analysis technique was applied to remove noise from noisy longitudinal series.

The results of this paper show that noise reduction is important for curve fitting in growth curve models (see the values of AIC and SIC in Tables 1 and 2). The results also show that using noise free series gives estimated parameters of growth curve models with smaller standard deviations than the noisy series (see Figures 2–4). Finally, we conclude that the singular

spectrum analysis technique can be used as a powerful tool for noise reduction in longitudinal measurements.

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Table 1: Fundamental statistics of parameters for noisy and noise free time series for boys.

Model	Parameter	Mean	Median	Minimum	Maximum	SD
Jeens-Bayely	α_1	69.81 (68.09)	71.04 (68.65)	47.49 (50.24)	93.42 (88.46)	7.80 (6.33)
	β_1	0.75 (0.77)	0.74 (0.77)	0.61 (0.66)	0.94 (0.86)	0.057 (0.037)
	γ_1	6.79 (6.67)	6.76 (6.68)	6.05 (5.87)	6.76 (6.98)	0.33 (0.22)
	δ_1	0.180 (0.189)	0.179 (0.189)	0.176 (0.188)	0.183 (0.191)	0.001 (0.0005)
$AIC = -0.32\,(-1.43),\ SIC = -0.145\,(-1.23)$						
Von Bertalanffy	α_2	103.3 (104.4)	105.19 (104.54)	80.82 (83.9)	130.09 (121.67)	10.39 (6.83)
	β_2	18.9 (18.3)	18.81 (18.31)	16.90 (17.37)	22.3 (19.42)	1.33 (0.56)
	γ_2	0.049 (0.045)	0.049 (0.044)	0.045 (0.042)	0.054 (0.048)	0.0027 (0.0013)
$AIC = 2.83\,(1.132),\ SIC = 2.96\,(1.43)$						
modify Count	α_3	50.88 (50.36)	50.0 (49.8)	38.1 (44.8)	67.6 (61.2)	5.73 (3.9)
	β_3	0.37 (0.36)	0.39 (0.37)	0.13 (0.25)	0.51 (0.48)	0.065 (0.039)
	γ_3	11.6 (11.8)	11.58 (11.8)	10.98 (11.38)	12.41 (12.17)	0.25 (0.13)
	δ_3	0.445 (0.427)	0.446 (0.427)	0.409 (0.402)	0.496 (0.454)	0.017 (0.008)
$AIC = -0.23\,(-1.36),\ SIC = -0.06\,(-1.14)$						

Table 2: Fundamental statistics of parameters for noisy and noise free time series for girls.

Model	Parameter	Mean	Median	Minimum	Maximum	SD
Jeens-Bayely	α_1	66.9 (69.38)	67.81 (69.43)	54.8 (64.7)	75.8 (73.6)	4.62 (1.94)
	β_1	0.76 (0.77)	0.76 (0.77)	0.61 (0.65)	0.97 (0.89)	0.054 (0.038)
	γ_1	6.46 (6.54)	6.47 (6.55)	5.78 (6.38)	7.16 (6.7)	0.26 (0.075)
	δ_1	0.160 (0.155)	0.161 (0.155)	0.147 (0.152)	0.173 (0.159)	0.004 (0.002)
$AIC = 0.12\,(-1.43),\ SIC = -0.311\,(-1.23)$						
Von Bertalanffy	α_2	103.8 (104.2)	104.1 (103.7)	85.57 (98.2)	117.1 (112.2)	6.55 (3.81)
	β_2	18.9 (19.1)	19.0 (19.2)	14.2 (17.8)	21.2 (20.2)	1.34 (0.66)
	γ_2	0.0449 (0.0451)	0.0449 (0.0451)	0.0427 (0.0442)	0.0473 (0.0460)	0.0009 (0.0004)
$AIC = 2.43\,(1.48),\ SIC = 2.57\,(1.63)$						
modified Count	α_3	49.7 (50.4)	50.13 (49.9)	35.8 (45.3)	60.2 (54.8)	4.57 (2.37)
	β_3	0.36 (0.35)	0.36 (0.35)	0.17 (0.30)	0.51 (0.42)	0.07 (0.03)
	γ_3	12.39 (12.25)	12.36 (12.20)	11.29 (10.9)	14.2 (13.2)	0.45 (0.35)
	δ_3	0.341 (0.341)	0.344 (0.339)	0.243 (0.312)	0.450 (0.382)	0.039 (0.014)
$AIC = -1.32\,(-2.12),\ SIC = -1.31\,(-2.04)$						